CONVECTION

- WE HAVE SEEN THAT HEAT TRANSFER FROM A SOLID TO A LIQUID IS GOVERNED BY NEWTON'S LAW OF COOLING:

$$q_{cv} = h_c A\left(t_w - t_f\right)$$

- UP TO NOW, WE HAVE SUPPOSED THAT , THE CONVECTION HEAT TRANSFER COEFFICIENT, h_c , WAS KNOWN.
- THE OBJECTIVES OF THIS CHAPTER ARE:
  - TO DISCUSS THE BASICS OF HEAT CONVECTION IN FLUIDS, AND
  - TO PRESENT METHODS TO PREDICT THE VALUE OF HEAT TRANSFER COEFFICIENT.
- CONVECTION IS THE TERM USED FOR HEAT TRANSFER IN A FLUID BECAUSE OF A COMBINATION OF:
  - CONDUCTION DUE TO MOLECULAR INTERACTIONS, AND
  - ENERGY TRANSPORT DUE TO THE MOTION OF THE FLUID BULK.
- THE MOTION OF THE FLUID BULK BRINGS THE HOT REGIONS IN CONTACT WITH THE COLD REGIONS.
- THE MOTION OF THE FLUID BULK MAY BE SUSTAINED:
  - BY A THERMALLY INDUCED DENSITY GRADIENT (NATURAL) CONVECTION), OR
  - BY A PRESSURE DIFFERENCE CREATED BY A PUMP (FORCED CONVECTION).

- IN BOTH CASES, THE DETERMINATION OF h_c REQUIRES THE KNOWLEDGE OF TEMPERATURE DISTRIBUTION IN THE FLUID FLOWING OVER THE HEATED WALL.
- SINCE THE FLUID IN THE VICINITY OF THE SOLID WALL IS PRACTICALLY MOTIONLESS, HEAT FLUX FROM THE WALL IS GIVEN BY:

$$q''_{cv} = -k_f \left(\frac{\partial t}{\partial y} \right)_w$$

$$q''_{cv} = h_c (t_w - t_f)$$

$$h_c = \frac{-k_f (\partial t / \partial y)_w}{t_w - t_f}$$

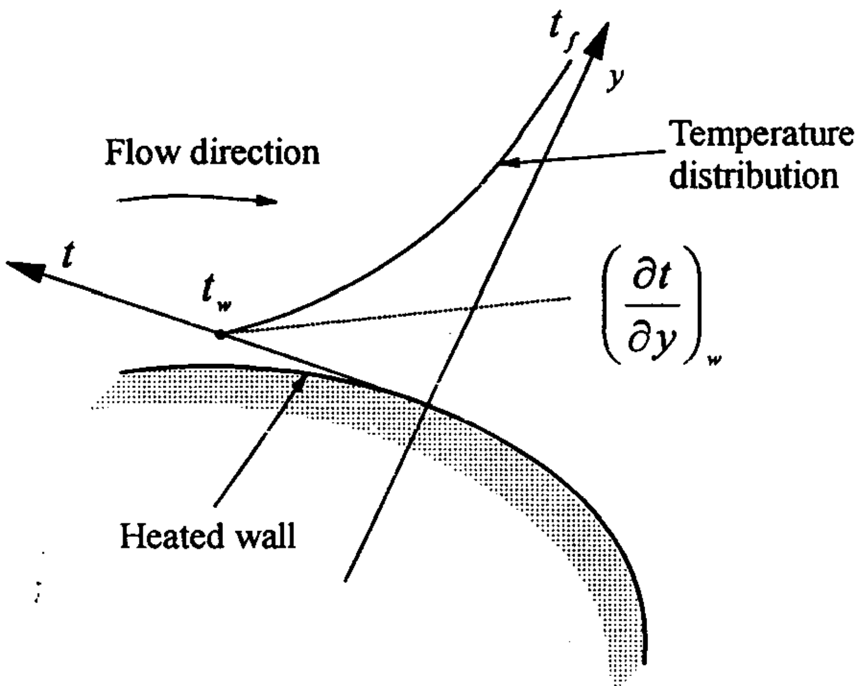

Figure 4.1 Variation of the temperature in the fluid next to the heated wall.

- ANALYTICAL DETERMINATION OF h_c REQUIRES THE SIMULTANEOUS SOLUTION OF:

  - MASS
  - MOMENTUM, AND
  - ENERGY

  CONSERVATION EQUATIONS.

- THE ANALYTICAL SOLUTION OF THESE EQUATIONS IS VERY DIFFICULT AND IT IS ONLY POSSIBLE FOR VERY SIMPLE CASES.

VISCOSITY

- THE NATURE OF VISCOSITY IS BEST UNDERSTOOD BY CONSIDERING A LIQUID PLACED BETWEEN TWO PLATES.

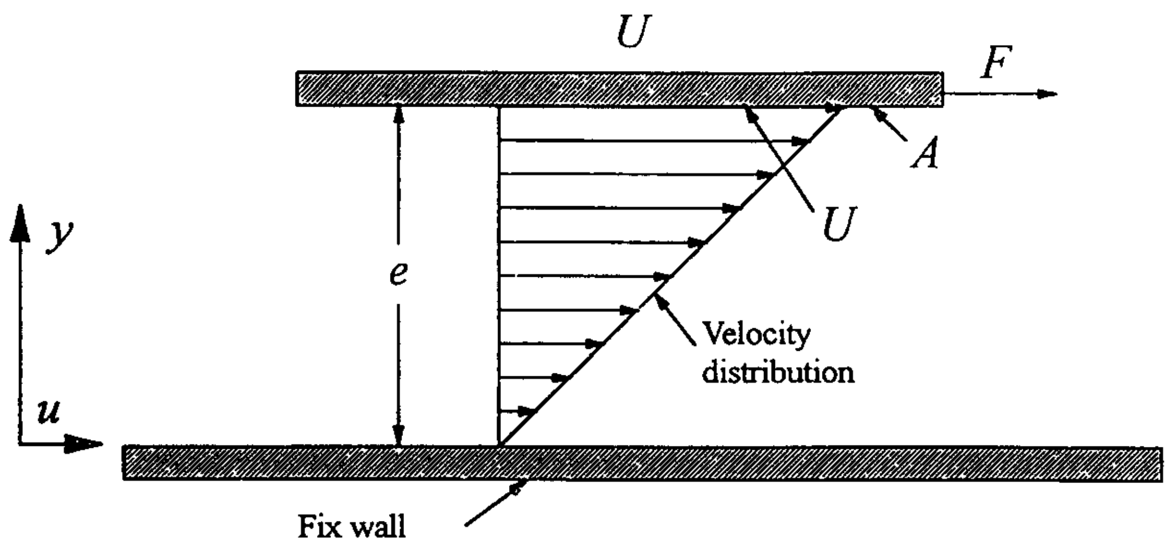

Figure 4.2 Shear stress applied to a fluid.

  - THE LOWER PLATE IS AT REST.
  - THE UPPER PLATE MOVES WITH A CONSTANT VELOCITY UNDER THE EFFECT OF A FORCE F.
  - THE DISTANCE BETWEEN THE PLATES IS SMALL.
  - THE SURFACE AREA OF THE UPPER PLATE IS: A.

- BECAUSE OF THE NON SLIP CONDITION ON THE WALLS THE FLUID VELOCITY:
  - AT THE LOWER PLATE IS ZERO,
  - AT THE UPPER PLATE IS U.

- UNDER THESE CONDITIONS, A LINEAR VELOCITY DISTRIBUTION DEVELOPS BETWEEN THE PLATES:

$$u = \frac{U}{e} y$$

- THE SLOPE:

$$\frac{du}{dy} = \frac{U}{e}$$

- THE SHEAR STRESS:

$$\tau = \frac{F}{A}$$

- IF THE FORCE F (or $\tau = F / A$) APPLIED TO THE UPPER PLATE CHANGES (i.e., UPPER PLATE VELOCITY), du / dy CHANGES AS:

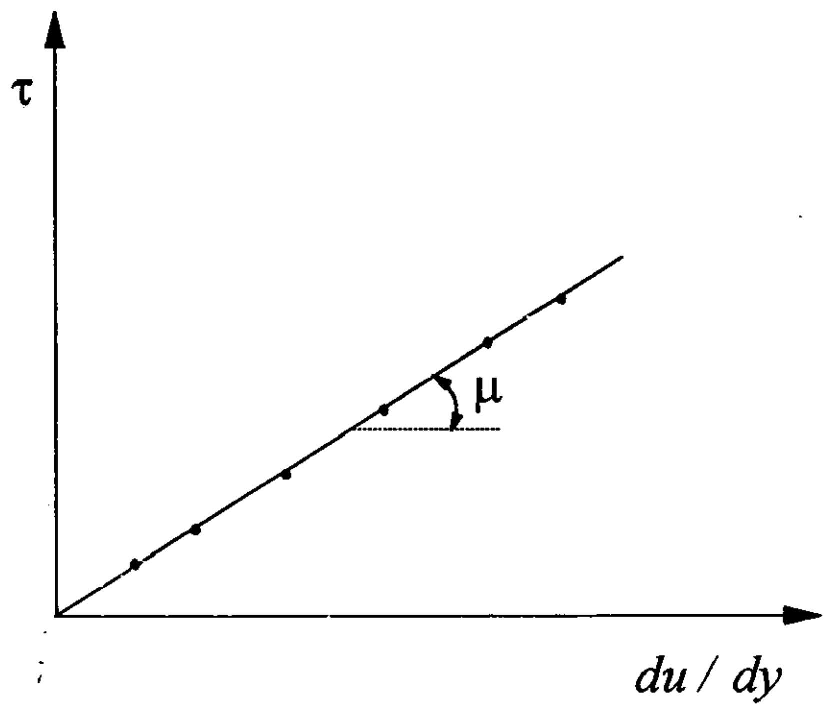

Figure 4.3 τ versus du/dy.

- WE SEE THAT:

$$\tau \sim \frac{du}{dy}$$

OR

$$\tau = \mu \frac{du}{dy}$$

- μ IS CALLED " THE DYNAMIC VISCOSITY."
- IN A MORE GENERAL WAY, CONSIDER A LAMINAR FLOW OVER A PLANE WALL.
- THE VELOCITY DISTRIBUTION HAS THE FOLLOWING FORM:

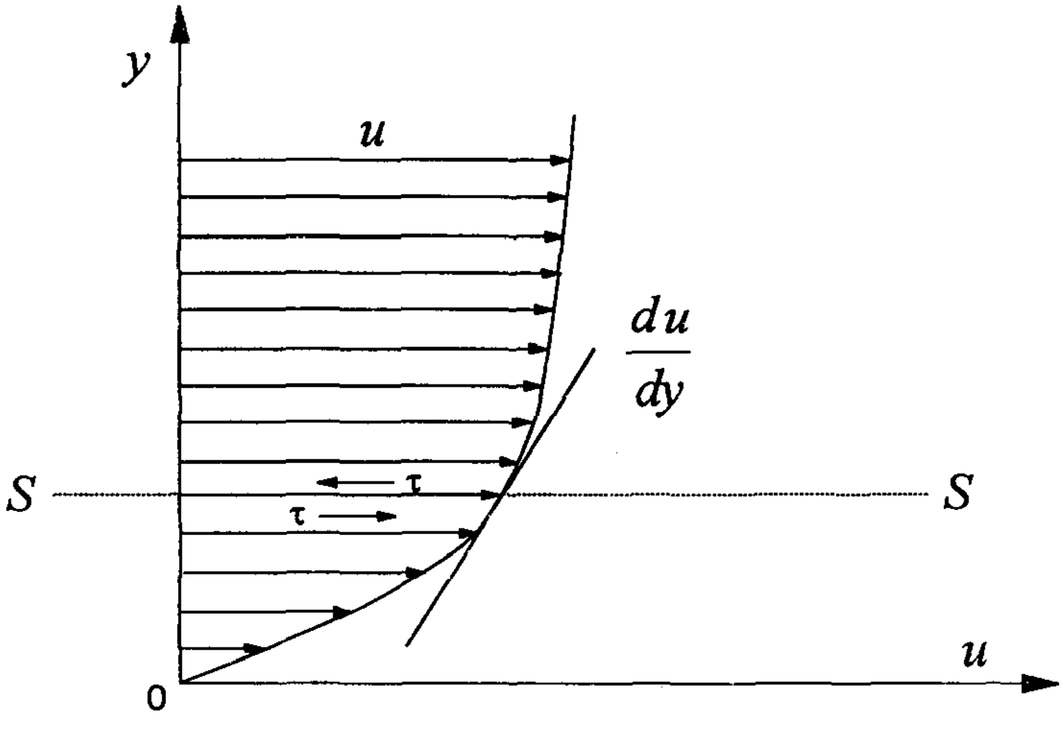

Figure 4.4 Velocity distribution next to a wall

- THIS DISTRIBUTION IS NOT LINEAR.
- SELECT A PLANE *SS* PARALLEL TO THE WALL.
- FLUID LAYERS ON EITHER SIDE OF *SS* EXPERIENCE A SHEARING FORCE DUE TO THEIR RELATIVE MOTION.
- THE SHEAR STRESS IS GIVEN BY:

$$\tau = \mu \left(\frac{du}{dy} \right)_{SS}$$

- THE RATIO OF THE DYNAMIC VISCOSITY TO THE SPECIFIC MASS:

$$\nu = \frac{\mu}{\rho}$$

IS CALLED "KINEMATIC VISCOSITY."

- UNITS:

$$\mu = \frac{\tau}{\frac{du}{dy}} = \frac{\mathrm{F}}{\mathrm{L}^2} \frac{\mathrm{T}}{\mathrm{L}} \mathrm{L} = \frac{\mathrm{FT}}{\mathrm{L}^2} \left(\frac{Ns}{m^2} \right)$$

$$\nu = \frac{\mu}{\rho} = \frac{\mathrm{L}^2}{\mathrm{T}} \left(\frac{m^2}{s} \right)$$

$$1\,N = 1\,kg \times 1 \frac{m}{s^2}$$

PHYSICAL BASIS OF THE VISCOSITY

- CONSIDER ONE DIMENSIONAL LAMINAR FLOW OF A DILUTE GAS ON A PLANE

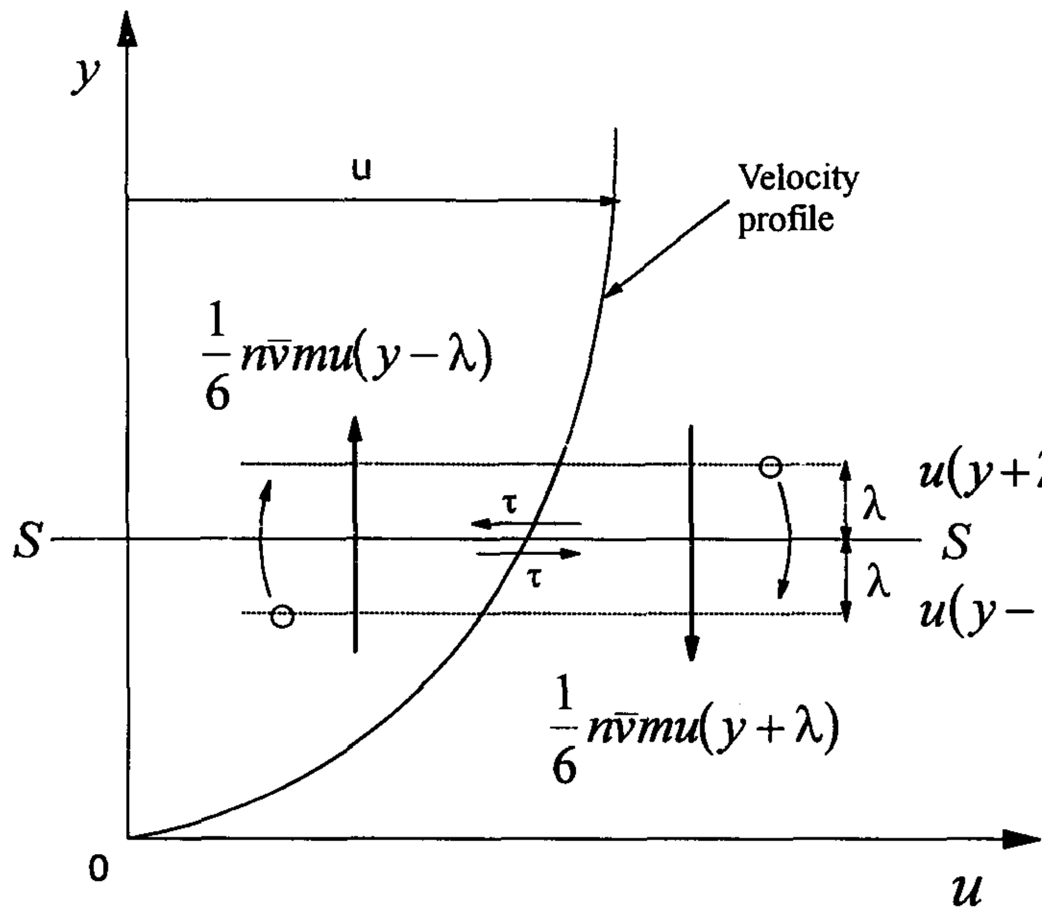

Figure 4.5 Flow of a dilute gas over a plane wall.

- $u = u(y)$
- CONSIDER A SURFACE *SS* PARALLEL TO THE WALL.
- BECAUSE OF THE *"RANDOM THERMAL VELOCITIES,"* GAS MOLECULES CROSS *SS* SURFACE BOTH
  - ABOVE, AND
  - BELOW.

- AT THE LAST COLLISION BEFORE CROSSING THE SURFACE *SS* EACH MOLECULE
  - ACQUIRE THE FLOW VELOCITY (u) CORRESPONDING TO THE HEIGHT AT WHICH THIS COLLISION TAKES PLACE.
- SINCE THE FLOW VELOCITY ABOVE THE PLANE *SS* IS GREATER THAN BELOW:
  - MOLECULES CROSSING FROM ABOVE TRANSPORT A GREATER MOMENTUM IN THE DIRECTION OF THE FLOW ACROSS THE SURFACE THAN
  - THAT TRANSPORTED BY THE MOLECULES CROSSING THE SAME SURFACE FROM BELOW.
- THE RESULT IS A NET MOMENTUM FLOW ACROSS THE PLANE *SS*
  - FROM THE REGION ABOVE
  - TO THE REGION BELOW.
- ACCORDING TO THE NEWTON'S SECOND LAW:
  - THE MOMENTUM CHANGE IN THE REGION ABOVE (OR BELOW) IS BALANCED BY THE "**VISCOUS FORCE.**"
- CONSEQUENTLY
  - THE REGION ABOVE *SS* IS SUBMITTED TO A FORCE DUE TO THE REGION BELOW ($-\tau$), AND
  - VICE VERSA (τ).

- BASED ON THE ABOVE DISCUSSION AN ESTIMATION OF THE "DYNAMIC VISCOSITY" CAN BE DONE:
  - IF THEY ARE n MOLECULES PER UNIT VOLUME OF THE DILUTE GAS, APPROXIMATELY:
    - (1/3) HAVE AVERAGE VELOCITY ($\bar{v}$) PARALLEL TO THE y-axis.
  - FROM THESE MOLECULES:
    - THE HALF ($\frac{n}{6}$) HAVE AN AVERAGE VELOCITY IN THE DIRECTION OF y^+, AND
    - THE OTHER HALF ($\frac{n}{6}$) IN THE DIRECTION OF y^-.
  - CONSEQUENTLY:
    - $\frac{n\bar{v}}{6}$ MOLECULES CROSS SS PER UNIT SURFACE AND UNIT TIME FROM ABOVE TO BELOW, AND
    - VICE VERSA.
  - MOLECULES COMING FROM ABOVE SS UNDERGO THEIR LAST COLLISION AT A DISTANCE EQUAL TO THE "MEAN **FREE PATH"** λ,
    - THEIR FLOW VELOCITY IS: $u(y+\lambda)$
    - THEIR MOMENTUM: $mu(y+\lambda)$
  - MOLECULES FROM BELOW:
    - VELOCITY: $u(y-\lambda)$
    - MOMENTUM: $mu(y-\lambda)$

- MOMENTUM COMPONENT IN THE **DIRECTION OF THE FLOW** THAT CROSSES THE SURFACE *SS* :

  - FROM ABOVE TO BELOW:

$$\frac{1}{6}n\bar{v}[mu(y+\lambda)]$$

  - FROM BELOW TO ABOVE:

$$\frac{1}{6}n\bar{v}[mu(y-\lambda)]$$

- THE NET MOMENTUM TRANSFER IS:

$$\frac{1}{6}n\bar{v}m[u(y-\lambda)-u(y+\lambda)]$$

- THE NET MOMENTUM TRANSFER SHOULD BE BALANCED BY THE VISCOUS FORCE τ .

$$\tau=\frac{1}{6}n\bar{v}m[u(y-\lambda)-u(y+\lambda)]$$

$$u(y+\lambda)\cong u(y)+\lambda\frac{du}{dy}$$

$$u(y-\lambda)\cong u(y)-\lambda\frac{du}{dy}$$

$$\tau=-\frac{1}{3}n\bar{v}m\lambda\frac{du}{dy}=-\mu\frac{du}{dy}$$

$$\mu=\frac{1}{3}n\bar{v}m\lambda$$

FLUID CONSERVATION EQUUATIONS - LAMINAR FLOW

OBJECTIVE:

DISCUSS THE BASIC ELEMENTS THAT ENTER IN THE ESTABLISHMENT OF THE CONSERVATION EQUATIONS FOR AN INCOMPRESSIBLE FLOW.

- LOCAL MASS CONSERVATION EQUATION

$$\frac{\partial}{\partial \tau}\rho + \vec{\nabla}.\rho\vec{v} = 0$$

FLUID IS INCOMPRESSIBLE
$\rho = const.$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- LOCAL MOMENTUM CONSERVATION EQUATION

$$\frac{\partial}{\partial \tau}\rho\vec{v} + \vec{\nabla}.\rho\vec{v}\,\vec{v} = -\vec{\nabla}.p\bar{\bar{I}} + \vec{\nabla}.\bar{\bar{\sigma}} + \rho\vec{g}$$

$\rho = const.$

$$\bar{\bar{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\bar{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} + \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\sigma_{zz} = 2\mu\frac{\partial w}{\partial z} + \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\sigma_{zx} = \sigma_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$\vec{v}\,\vec{v}$: diadic product of two vectors.

$\bar{\bar{I}}$: unit tensor.

$\bar{\bar{\sigma}}$: stress tensor.

$\vec{g}$:acceleration of the gravity.

x - COMPONENT

$$\rho\left(\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

y - COMPONENT

$$\rho\left(\frac{\partial v}{\partial \tau}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

z - COMPONENT

$$\rho\left(\frac{\partial w}{\partial \tau}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

THESE EQUATIONS ARE KNOWN AS "NAVIER-STOKES" EQUATIONS.

- LOCAL ENERGY CONSERVATION EQUATION

$$\frac{\partial}{\partial \tau}\rho\left(h+\frac{1}{2}\vec{v}.\vec{v}\right)-\frac{\partial}{\partial \tau}p+\vec{\nabla}.\rho\left(h+\frac{1}{2}\vec{v}.\vec{v}\right)\vec{v}$$
$$=-\vec{\nabla}.\vec{q}''+\vec{\nabla}.\left(\bar{\bar{\sigma}}.\vec{v}\right)+\rho\,\vec{g}.\vec{v}+q'''$$

ENTHALPY FORM

$$\bar{\bar{\sigma}}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}$$

$$\vec{q}''=-k_f\vec{\nabla}.t$$

- KINETIC ENERGY NEGLIGIBLE.
- POTENTIAL ENERGY NELIGIBLE.
- c_p, ρ, μ, k_f ARE CONSTANT.
- PRESSURE DOESNOT CHANGE WITH TIME:
  $$\frac{\partial p}{\partial \tau}=0$$
- NO ENERGY GENERATION
  $$\dot{Q}_g=0$$

$$\rho c_p\left(\frac{\partial t}{\partial \tau}+u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}+w\frac{\partial t}{\partial z}\right)=k_f\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right)+\mu\phi$$

WHERE

$$\phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2$$

- THE SOLUTION OF THE ENERGY EQUATION IN CONJUNCTION WITH

  - CONTINUITY EQUATION,
  - NAVIER-STOKES (MOMENTUM) EQUATIONS, AND
  - APPROPRIATE BOUNDARY CONDITIONS

  YIELDS THE TEMPERATURE DISTRIBUTION IN THE FLUID OVER THE HEATED WALL.

- FOR AN INCOMPRESSIBLE FLOW, THE UNKNOWNS ARE:

$$u, v, w, p, t$$

- THERE ARE 5 EQUATIONS TO DETERMINE THESE UNKNOWNS.

- ONCE THE TEMPERATURE DISTRIBUTION IN THE FLUID WASHING THE HEATED WALL IS KNOWN, THE CONVECTION HEAT TRANSFER COEFFICIENT IS DETERMINED BY:

$$h_c = \frac{-k_f(\partial t / \partial y)_w}{t_w - t_f}$$

- THE CONSERVATION EQUATIONS ARE NONLINEAR.

- NO GENERAL METHODS EXIST FOR THEIR SOLUTION.

- ANALYTICAL SOLUTIONS ARE LIMITED TO VERY SIMPLE CASES.

- FORTUNATELY, A LARGE NUMBER OF ENGINEERING PROBLEMS CAN BE HANDLED:
  - BY USING ONE DIMENSIONAL MODELS, AND
  - EXPERIMENTALLY DETERMINED CONSTITUTIVE EQUATIONS.
- THE SOLUTIONS CAN BE OBTAINED MORE EASILY.

- THE ABOVE CONSERVATION EQUATIONS APPLY ONLY TO LAMINAR FLOWS.
  - IN A LAMINAR FLOW, FLUID PARTICLES FOLLOW WELL DEFINED STEAMLINES.
  - THE STREAMLINES REMAIN PARALLEL TO EACH OTHER AND THEY ARE SMOOTH.
  - HEAT AND MOMENTUM ARE TRANSFERRED ACROSS THE STEAMLINES ONLY BY MOLECULAR DIFFUSION.
  - LAMINAR FLOWS EXIST AT LOW VELOCITIES.

TURBULENT FLOW

- IN TURBULENT FLOW, THE FLOW PARAMETERS:

  - VELOCITY
  - PRESSURE
  - TEMPERATURE

FLUCTUATE ABOUT A MEAN VALUE.

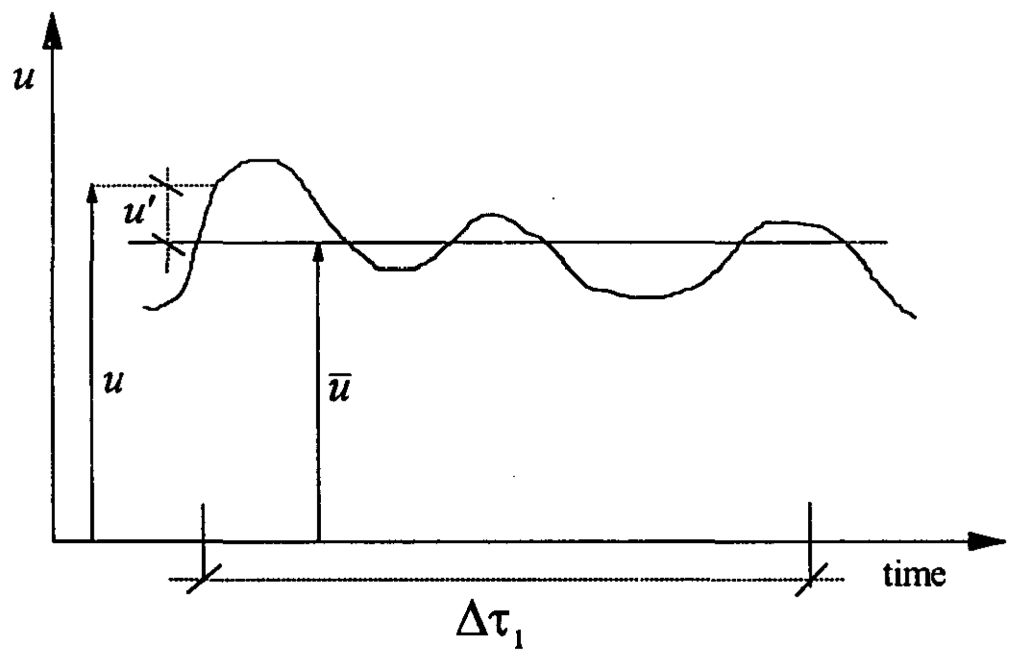

Figure 4.6 Turbulent velocity fluctuations about a time average.

- INSTANTANEOUS VALUE OF FLOW PARAMETERS (*u, v, w, p, t*) ARE WRITTEN AS:

$$u = \overline{u} + u'$$
$$v = \overline{v} + v'$$
$$w = \overline{w} + w'$$
$$p = \overline{p} + p'$$
$$t = \overline{t} + t'$$

$\overline{u}, \overline{v}, \overline{w}, \overline{p}, \overline{t}$: TIME AVERAGE FLOW PARAMETERS,
u', v', w', p', t' : TIME DEPENDENT FLOW PARAMETERS.

- BECAUSE OF THE RANDOMLY FLUCTUATING VELOCITIES, THE FLUID PARTICLES DO NOT STAY IN ONE LAYER (OR STREAMLINE) AND FOLLOWS A TORTUOUS PATH.
- CONSEQUENTLY, A MIXING OCCURS BETWEEN FLUID LAYERS AND THIS INCREASES THE HEAT AND MOMENTUM EXCHANGES.
- THE AVERAGE OF A FLOW PARAMETER IS GIVEN BY:

$$\bar{f} = \frac{1}{\Delta\tau_1}\int_0^{\Delta\tau_1} f\,d\tau$$

INDEPENDENT OF TIME FOR STEADY FLOW

- THE TIME INTERVAL, $\Delta\tau_1$, SHOULD BE LARGE TO EXCEED AMPLY THE PERIOD OF THE FLUCTUATIONS.
- THE TIME AVERAGE OF f':

$$\bar{f}' = \frac{1}{\Delta\tau_1}\int_0^{\Delta\tau_1} f'\,d\tau = \frac{1}{\Delta\tau_1}\int_0^{\Delta\tau_1} (f - \bar{f})\,d\tau = \bar{f} - \bar{f} = 0$$

CONSERVATION EQUATIONS FOR STEADY TURBULENT FLOW

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\rho\left(\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v}{\partial \tau}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial w}{\partial \tau}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

$$\rho c_p\left(\frac{\partial t}{\partial \tau}+u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}+w\frac{\partial t}{\partial z}\right)=k_f\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right)$$

WHERE VISCOUS DISSIPATION TERM, $\mu\phi$, IS IGNORED.

$$u=\overline{u}+u'$$
$$v=\overline{v}+v'$$
$$w=\overline{w}+w'$$
$$p=\overline{p}+p'$$
$$t=\overline{t}+t'$$

CONSERVATION EQUATIONS FOR STEADY TURBULENT FLOW

LAMINAR FLOW CONSERVATION EQUATIONS

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$
$$p = \bar{p} + p'$$
$$t = \bar{t} + t'$$

AVERAGING RULES

$$f = \bar{f} + f'$$

$$g = \bar{g} + g'$$

$$\overline{f'} = \overline{g'} = 0$$

$$\overline{f + g} = \bar{f} + \bar{g}$$

$$\overline{\bar{f} f'} = \overline{\bar{g} g'} = 0$$

$$\overline{fg} = \bar{f}\,\bar{g} + \overline{f' g'}$$

$$\overline{f^2} = (\bar{f})^2 + \overline{(f')^2}$$

$$\overline{\left(\frac{\partial f}{\partial x}\right)} = \frac{\partial \bar{f}}{\partial x}$$

$$\frac{\partial \bar{f}}{\partial \tau} = 0$$

$$\overline{\left(\frac{\partial f}{\partial \tau}\right)} = 0$$

$$\overline{cf} = c\bar{f}$$

$$c = const.$$

CONSERVATION EQUATIONS FOR STEADY TURBULENT FLOW

- MASS CONSERVATION EQUATION

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

- MOMENTUM CONSERVATION EQUATIONS

$$\rho\left(\frac{\partial \bar{u}}{\partial \tau}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}\boxed{-\frac{\partial}{\partial x}\rho\overline{u'^2}-\frac{\partial}{\partial y}\rho\overline{u'v'}-\frac{\partial}{\partial z}\rho\overline{u'w'}}$$

$$\rho\left(\frac{\partial \bar{v}}{\partial \tau}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}\boxed{-\frac{\partial}{\partial x}\rho\overline{u'v'}-\frac{\partial}{\partial y}\rho\overline{v'^2}-\frac{\partial}{\partial z}\rho\overline{v'w'}}$$

$$\rho\left(\frac{\partial \bar{w}}{\partial \tau}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}\boxed{-\frac{\partial}{\partial x}\rho\overline{u'w'}-\frac{\partial}{\partial y}\rho\overline{v'w'}-\frac{\partial}{\partial z}\rho\overline{w'^2}}$$

Body forces ignored

- ENERGY CONSERVATION EQUATION

$$\rho c_p\left(\frac{\partial \bar{t}}{\partial \tau}+u\frac{\partial \bar{t}}{\partial x}+v\frac{\partial \bar{t}}{\partial y}+w\frac{\partial \bar{t}}{\partial z}\right)=k\nabla^2\bar{t}\boxed{-\frac{\partial}{\partial x}\rho c_p\overline{u't'}-\frac{\partial}{\partial y}\rho c_p\overline{v't'}-\frac{\partial}{\partial z}\rho c_p\overline{w't'}}$$

$$\boxed{\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial z^2}}$$

- WHEN TURBULENT FLOW EQUATIONS ARE COMPARED WITH STEADY STATE LAMINAR FLOW EQUATIONS WE OBSERVE ADDITIONAL TERMS (FRAMED WITH DOTTED LINES).

- THESE TERMS ARE ASSOCIATED WITH TURBULENT FLUCTUATIONS.

- IN THE MOMENTUM EQUATIONS THESE ADDITIONAL (FLUCTUATING) TERMS REPRESENTS "**TURBULENT MOMENTUM FLUX,**" WHICH ARE USUALLY REFERRED TO AS:

APPARENT STRESSES, OR REYNOLDS' STRESSES.

- IN THE ENERGY EQUATION THE FLUCTUATING TERMS REPRESENT:

THE COMPONENTS OF THE TURBULENT ENERGY FLUX.

CONCEPT OF BOUNDARY LAYER

- CONSIDER A VISCOUS FLOW OVER A PLATE

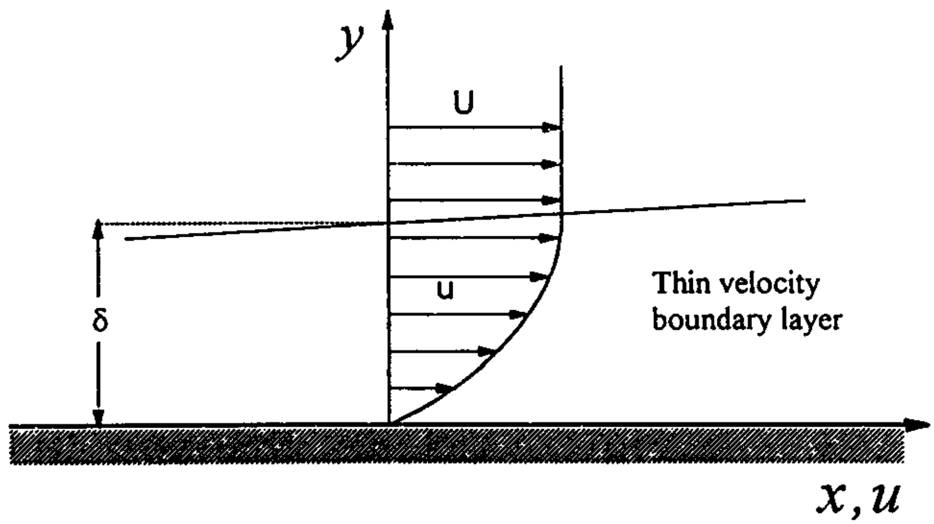

Figure 4.7 Velocity profile in the vicinity of a plate

- THE VELOCITY OF THE FLUID CLOSE TO THE PLATE VARIES FROM ZERO TO THE VELOCITY OF THE FREE STREAM *U*.
- BECAUSE OF THE VELOCITY GRADIENT, THERE ARE VISCOUS STRESSES IN THIS REGION.
- THE MAGNITUDE OF THE VISCOUS STRESSES INCREASES AS WE GET CLOSER TO THE WALL.

- BASED ON THE ABOVE OBSERVATION, PRANDLT

  - FOR SMALL VISCOSITY FLUIDS, AND
  - LARGE VELOCITIES

  DIVIDED THE FLOW ON THE WALL INTO TWO REGIONS:

  - ► A VERY THIN LAYER (BOUNDARY LAYER) IN THE IMMEDIATE NEIGHBOR OF THE WALL IN WHICH THE VELOCITY INCREASES RAPIDLY WITH THE DISTANCE TO THE WALL, i.e., THERE ARE:

    - HIGH GRADIENTS
    - HIGH SHEAR STRESSES.

  - ► A POTENTIAL FLOW REGION, OUTSIDE OF THE BOUNDARY LAYER, WHERE THERE IS ALMOST NO VELOCITY GRADIENT, i.e., NO VISCOUS STRESS.

- THE LIMIT OF THE BOUNDARY LAYER (BOUNDARY LAYER THICKNESS, DENOTED BY δ) IS

  "THE DISTANCE FROM THE WALL WHERE THE FLOW VELOCITY REACHES 99% OF THE FREE STREAM VELOCITY."

- A BOUNDARY LAYER CAN BE:

  - ► LAMINAR, OR
  - ► TURBULENT.

LAMINAR BOUNDARY LAYER

- FLOW IN THE BOUNDARY LAYER IS LAMINAR WHEN THE FLUID PARTICLES MOVE ALONG THE STREAM LINES IN AN ORDERLY MANNER.

- THE CRITERION FOR A FLOW OVER A FLAT PLATE TO BE LAMINAR IS:

$$Re_x = \frac{\rho U x}{\mu} < 5 \times 10^5$$

- THE ANALYSIS OF THE BOUNDARY LAYER CAN BE CONDUCTED BY USING:

  1. LOCAL MASS, MOMENTUM AND ENERGY CONSERVATION EQUATIONS, OR

  2. AN APPROXIMATE METHOD BASED ON THE INTEGRAL CONSERVATION EQUATIONS OF MASS, MOMENTUM AND ENERGY.

LAMINAR BOUNDARY LAYER CONSERVATION EQUATIONS - LOCAL FORMULATION

- MASS AND MOMENTUM EQUATION

  CONSIDER THE FLOW (AND HEAT TRANSFER) ON A FLAT PLATE ILLUSTRATED BELOW:

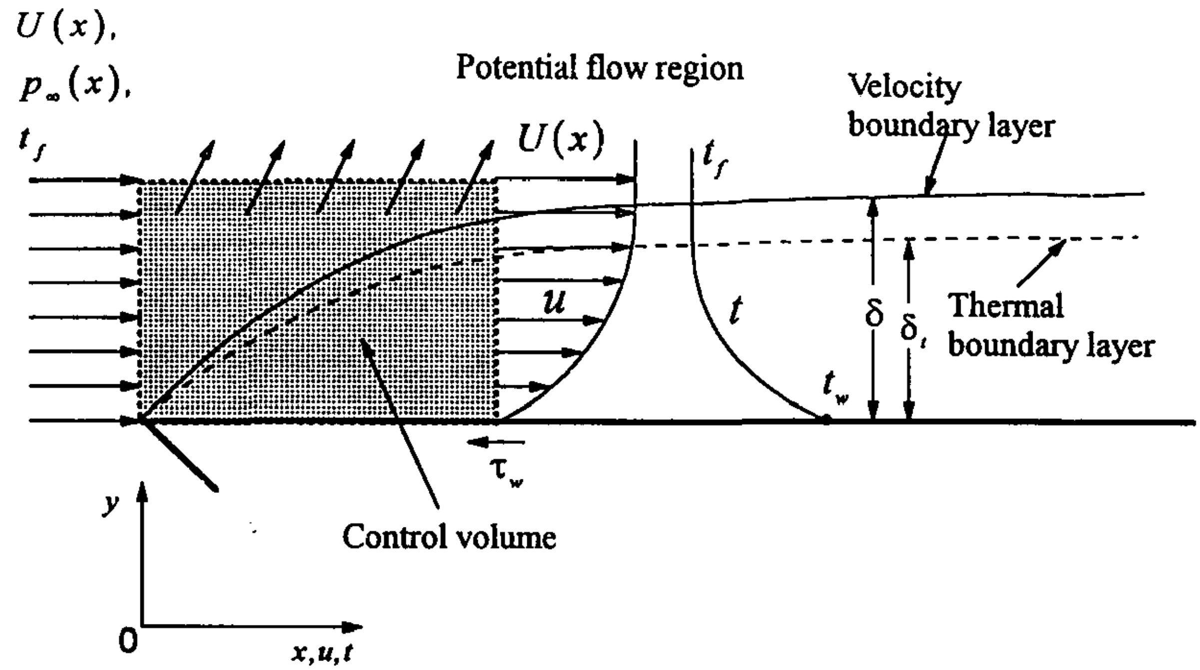

Figure 4.8 Velocity and thermal boundary layers in a laminar flow on a flat plate

MASS AND MOMENTUM CONSERVATION EQUATIONS FOR A LAMINAR FLOW

- STEADY STATE FLOW.
- TWO DIMENSIONAL FLOW (NO VELOCITY AND TEMPERATURE GRADIENTS IN THE z-DIRECTION.
- NO BODY FORCES.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

THESE EQUATIONS ARE NON-LINEAR

- AN ORDER OF MAGNITUDE ANALYSIS SHOWS THAT:

$$\nu\frac{\partial^2 u}{\partial u^2},\ u\frac{\partial v}{\partial x},\ v\frac{\partial v}{\partial y}$$

$$\nu\frac{\partial^2 v}{\partial x^2},\ \nu\frac{\partial^2 v}{\partial y^2}$$

ARE VERY SMALL AND CAN BE IGNORED.

MASS AND MOMENTUM CONSERVATION

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

BOUNDARY CONDITIONS

$y = 0 \quad u = v = 0$

$y = \infty \quad u = U(x)$

THE SOLUTION OF THE ABOVE SYSTEM OF EQUATIONS YIELDS THE VELOCITY DISTRIBUTION AND THE BOUNDARY LAYER THICKNESS

- $-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0$

  SHOWS THAT, AT A GIVEN x, THE PRESSURE IS CONSTANT IN THE y-DIRECTION, i.e., IT IS INDEPENDENT OF y.
- THE SOLUTION OF THIS SYSTEM OF EQUATIONS IS BEYOND THE SCOPE OF THIS COURSE.
- CERTAIN PARTICULARITIES OF THESE EQUATIONS WILL BE USED LATTER TO DISCUSS THE THICKNESS OF THE VELOCITY BOUNDARY LAYER.

- BERNOULLI EQUATION FOR THE POTENTIAL FLOW REGION

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

IN THE POTENTIAL REGION:

$$u=U(x)$$

$$\frac{\partial u}{\partial y}=0$$

$$\frac{\partial^2 u}{\partial y^2}=0$$

$$U(x)\frac{\partial U(x)}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

INTEGRATION

$$p(x)+\frac{1}{2}\rho U^2(x)=const.$$

THIS IS THE BERNOULLI EQUATION.

- ENERGY CONSERVATION EQUATION

  - IF $t_w \neq t_f$ A THERMAL BOUNDARY LAYER FORMS ON THE PLATE.

  - THROUGH THIS LAYER, THE FLUID TEMPERATURE MAKES THE TRANSITION FROM THE WALL TEMPERATURE, t_w , TO THE FREE STREAM TEMPERATURE, t_f .

  - THE LIMIT OF THE THERMAL BOUNDARY LAYER (BOUNDARY LAYER THICKNESS, δ_t) IS THE DISTANCE FROM THE WALL WHERE THE FLOW TEMPERATURE REACHES 99% OF THE FREE STREAM TEMPERATURE.

  - THE THICKNESS OF THE THERMAL BOUNDARY LAYER IS IN THE SAME ORDER OF MAGNITUDE OF THE THICKNESS OF THE VELOCITY BOUNDARY LAYER.

  - HOWEVER, THEY ARE NOT NECESSARILY EQUAL.

ENERGY CONSERVATION EQUATION

$$\rho c_p\left(\frac{\partial t}{\partial \tau}+u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}+w\frac{\partial t}{\partial z}\right)=k\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right)+\mu\phi$$

- TWO DIMENSIONAL FLOW.
- STEADY STATE FLOW.
- VISCOUS DISSIPATION NEGLECTED COMPARED TO THE WALL HEAT FLUX:

$$\mu\phi \approx 0$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}\right)$$

- AN ORDER OF MAGNITUDE ANALYSIS SHOWS THAT

$$\frac{\partial^2 t}{\partial x^2}$$

IS SMALL AND CAN BE IGNORED.

$$\alpha=\frac{k}{c_p\rho}$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\alpha\frac{\partial^2 t}{\partial y^2}$$

ENERGY CONSERVATION EQUATION

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \alpha\frac{\partial^2 t}{\partial y^2}$$

- BOUNDARY CONDITIONS FOR A CONSTANT WALL TEMPERATURE:

  $y = 0 \quad t = t_w$

  $y = \infty \quad t = t_f$

  $x = 0 \quad t = t_f$

THE INTEGRATION OF THIS EQUATION IN CONJUNCTION WITH MASS AND MOMENTUM EQUATIONS YIELDS THE TEMPERATURE DISTRIBUTION AND THE THICKNESS OF THE THERMAL BOUNDARY LAYER.

CONSERVATION EQUATIONS - INTEGRAL FORMULATION

- THE OBJECTIVE OF THE STUDY OF A BOUNDARY LAYER IS TO DETERMINE ON THE WALL:
  - THE SHEAR FORCES, AND
  - THE HEAT TRANSFER COEFFICIENT.
- THE SOLUTION OF THE GOVERNING EQUATIONS WE HAVE JUST DISCUSSED TO OBTAIN THE ABOVE QUANTITIES IS QUITE DIFFICULT AND IS NOT WITHIN THE SCOPE OF THIS COURSE.
- WE WILL DISCUSS NOW A SIMPLE APPROACH CALLED "THE INTEGRAL METHOD:"
  - TO ANALYZE THE BOUNDARY LAYER, AND
  - TO DETERMINE THE SHEAR STRESSES AND THE HEAT TRANSFER COEFFICIENT.
- THIS METHOD WAS INTRODUCED BY " von KARMAN" IN 1947.
- INTEGRAL METHOD CONSISTS OF FIXING THE ATTENTION ON THE OVER-ALL BEHAVIOR OF THE BOUNDARY LAYER INSTEAD OF THE LOCAL BEHAVIOR OF THE SAME LAYER.
- TO DERIVE THE INTEGRAL BOUNDARY LAYER EQUATIONS, THE INTEGRAL CONSERVATION EQUATIONS (CHAPTER 2) WILL BE APPLIED:
  - TO A FIX CONTROL VOLUME
  - UNDER STEADY STATE CONDITIONS.

CONSERVATION EQUATIONS - INTEGRAL FORMULATION

- BOUNDARY LAYER INTEGRAL MASS CONSERVATION EQUATION.

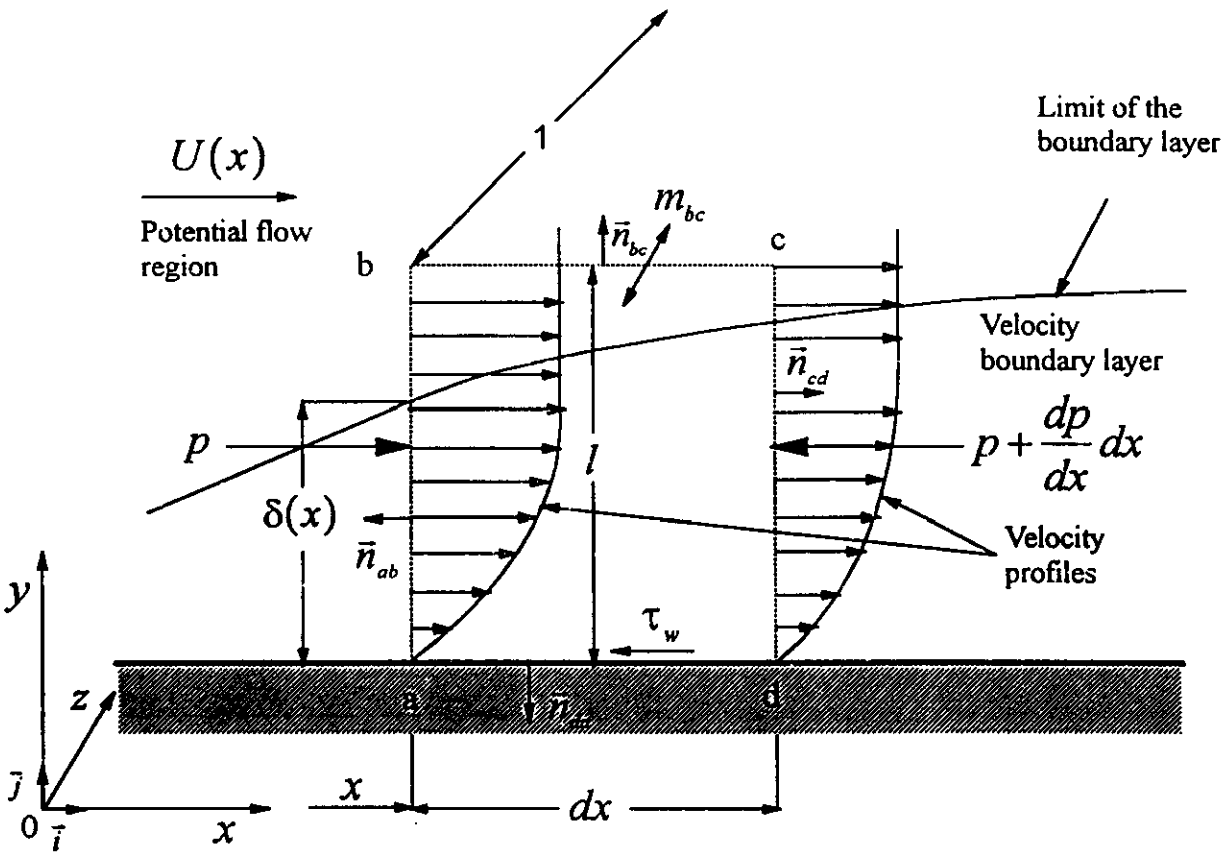

Figure 4.9 Control volume for approximate analysis of the velocity boundary layer

MASS CONSERVATION EQUATION - INTEGRAL FORMULATION

$$\frac{d}{d\tau}\int_{V(\tau)} \rho \, dV = -\int_{A(\tau)} \vec{n}.\rho(\vec{v}-\vec{\omega})\,dA$$

- FIX CONTROL VOLUME $\vec{\omega} = 0$
- STEADY STATE

$$\int_A \vec{n}.\rho\vec{v}\,dA = 0$$

APPLICATION TO THE SELECTED CONTROL VOLUME

$$\int_A \vec{n}.\rho\vec{v}\,dA = \dot{m}_{ab} + \dot{m}_{bc} + \dot{m}_{cd} + \dot{m}_{da} = 0$$

$$\dot{m}_{ab} = \int_{A_{ab}} \vec{n}_{ab}.\rho u\vec{i}\,dA = -\left[\int_0^l \rho u dy\right]_x$$

$$\dot{m}_{cd} = \int_{A_{cd}} \vec{n}_{cd}.\rho u\vec{i}\,dA = \left[\int_0^l \rho u dy\right]_{x+dx}$$

or

$$\dot{m}_{cd} = \left[\int_0^l \rho u dy\right]_x + \frac{d}{dx}\left[\int_0^l \rho u dy\right]_x dx$$

$$A_{ab} = A_{cd} = l \times 1$$

$\dot{m}_{da} = 0$, *solid wall*

$$\dot{m}_{bc} = -\frac{d}{dx}\left[\int_0^l \rho u dy\right]_x dx$$

CONSERVATION EQUATIONS - INTEGRAL FORMULATION

- BOUNDARY LAYER MOMENTUM CONSERVATION EQUATION

$$\frac{d}{d\tau}\int_{V(\tau)} \rho \vec{v}\, dV = -\int_{A(\tau)} \vec{n}.\rho \vec{v}(\vec{v}-\vec{\omega})\, dA - \int_{A(\tau)} \vec{n}.p\bar{\bar{I}}\, dA + \int_{A(\tau)} \vec{n}.\bar{\bar{\sigma}}\, dA + \int_{V(\tau)} \rho \vec{g}\, dV$$

- ► STEADY STATE
- ► $\vec{\omega} = 0$
- ► GRAVITY NEGLECTED

$$-\int_{A} \vec{n}.\rho \vec{v}\vec{v}\, dA - \int_{A} \vec{n}.p\bar{\bar{I}}\, dA + \int_{A} \vec{n}.\bar{\bar{\sigma}}\, dA = 0$$

- ► NO PRESSURE VARIATION IN THE y-DIRECTION.
- ► μ IS CONSTANT.
- ► STRESS FORCES ACTING ON ALL FACES EXCEPT THE FACE da ARE NEGLIGIBLE.

$$-M_{ab} - M_{cd} - M_{bc} - \int_{A_{ab}} \vec{n}_{ab}.p_x\bar{\bar{I}}\, dA - \int_{A_{cd}} \vec{n}_{cd}.p_{x+dx}\bar{\bar{I}}\, dA + \int_{A_{da}} \vec{n}_{da}.\bar{\bar{\sigma}}\, dA = 0$$

M MOMENTUM

MOMENTUM CONSERVATION EQUATION - INTEGRAL FORMULATION

$$-M_{ab} - M_{cd} - M_{bc} - \int_{A_{ab}} \vec{n}_{ab} \cdot p_x \bar{\bar{I}} dA - \int_{A_{cd}} \vec{n}_{cd} \cdot p_{x+dx} \bar{\bar{I}} dA + \int_{A_{da}} \vec{n}_{da} \cdot \bar{\bar{\sigma}} dA = 0$$

$$M_{ab} = \int_{A_{ab}} \vec{n}_{ab} \cdot \rho(u\vec{i})(u\vec{i}) dA = \vec{n}_{ab} \left[\int_0^l \rho u^2 dy \right]_x$$

$$M_{cd} = \int_{A_{cd}} \vec{n}_{cd} \cdot \rho(u\vec{i})(u\vec{i}) dA = \vec{n}_{cd} \left[\int_0^l \rho u^2 dy \right]_{x+dx}$$

or

$$M_{cd} = \vec{n}_{cd} \left[\int_0^l \rho u^2 dy \right]_x + \vec{n}_{cd} \frac{d}{dx} \left[\int_0^l \rho u^2 dy \right]_x dx$$

$$M_{bc} = \dot{m}_{bc} U(x)\vec{i} = -U(x)\vec{i} \frac{d}{dx} \left[\int_0^l \rho u dy \right]_x dx$$

$$\int_{A_{ab}} \vec{n}_{ab} \cdot p_x \bar{\bar{I}}. dA = \vec{n}_{ab} p_x l$$

$$\int_{A_{cd}} \vec{n}_{cd} \cdot p_{x+dx} \bar{\bar{I}}. dA = \vec{n}_{cd} p_{x+dx} l$$

or

$$\int_{A_{cd}} \vec{n}_{cd} \cdot p_{x+dx} \bar{\bar{I}}. dA = \vec{n}_{cd} (p_x + \frac{dp_x}{dx} dx) l$$

$$\int_{A_{da}} \vec{n}_{da} . \bar{\bar{\sigma}} dA = -\tau_w \vec{i}\, dx$$

$$A_{ab} = A_{cd} = l \times 1$$

$$A_{bc} = A_{da} = dx \times 1$$

MOMENTUM CONSERVATION EQUATION - INTEGRAL FORMULATION

$$-\vec{n}_{ab}\left[\int_0^l \rho u^2 dy\right]_x - \vec{n}_{cd}\left[\int_0^l \rho u^2 dy\right]_x - \vec{n}_{cd}\frac{d}{dx}\left[\int_0^l \rho u^2 dy\right]_x dx$$

$$+U(x)\vec{i}\,\frac{d}{dx}\left[\int_0^l \rho u dy\right]_x dx - \vec{n}_{ab} p_x l - \vec{n}_{cd}\left(p_x + \frac{dp_x}{dx}dx\right)l - \tau_w \vec{i}\,dx = 0$$

$\times \vec{i}$

KNOWING THAT

$\vec{n}_{ab}.\vec{i} = -1$

$\vec{n}_{cd}.\vec{i} = 1$

$$-\frac{d}{dx}\int_0^l \rho u^2 dy + U(x)\frac{d}{dx}\int_0^l \rho u dy = \frac{dp}{dx} l + \tau_w$$

ADDING AND SUBTRACTING TO THE LEFT SIDE

$$\frac{dU(x)}{dx}\left[\int_0^l \rho u dy\right]$$

$$\rho\frac{d}{dx}\int_0^l (U(x) - u) u dy - \rho\frac{dU(x)}{dx}\int_0^l u dy = \frac{dp}{dx} l + \tau_w$$

MOMENTUM CONSERVATION EQUATION - INTEGRAL FORMULATION

$$\rho\frac{d}{dx}\int_0^l (U(x)-u)u\,dy-\rho\frac{dU(x)}{dx}\int_0^l u\,dy=\frac{dp}{dx}l+\tau_w$$

USING THE BERNOULLI EQUATION:

$$p(x)+\frac{1}{2}\rho U^2(x)=const.$$

$$\frac{dp}{dx}=-\rho U(x)\frac{dU(x)}{dx}$$

$$l\frac{dp}{dx}=\int_0^l\frac{dp}{dx}dy$$

$$=-\int_0^l\rho U(x)\frac{dU(x)}{dx}dy$$

$$\rho\frac{d}{dx}\int_0^{\delta(x)}(U(x)-u)u\,dy+\rho\frac{dU(x)}{dx}\int_0^{\delta(x)}(U(x)-u)\,dy=\tau_w$$

$$l\to\delta(x)$$

$$\tau_w=\mu\left(\frac{du}{dx}\right)_{y=0}$$

THIS IS THE INTEGRAL MOMENTUM EQUATION OF A STEADY, LAMINAR AND INCOMPRESSIBLE BOUNDARY LAYER.

CONSERVATION EQUATION - INTEGRAL FORMULATION

- BOUNDARY LAYER ENERGY COSERVATION EQUATION

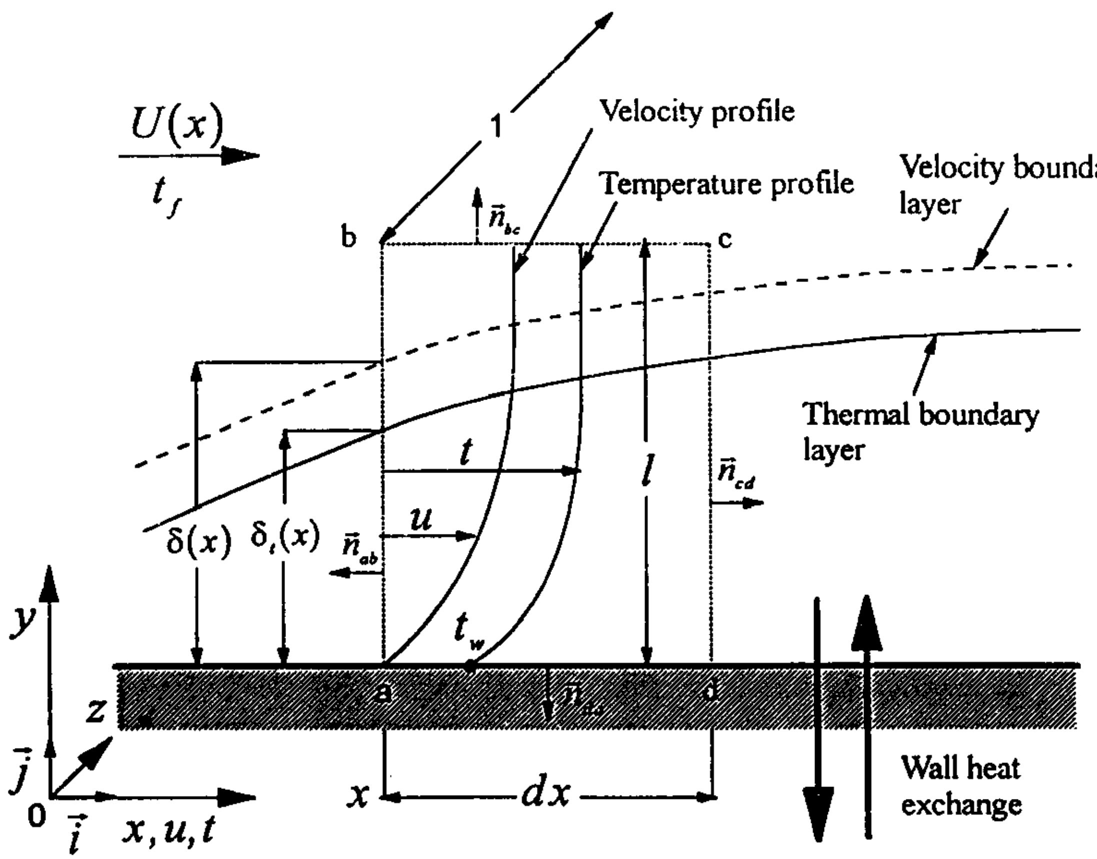

Figure 4.10 Control volume for integral conservation of energy

ENERGY CONSERVATION EQUATION - INTEGRAL FORMULATION

$$\frac{d}{d\tau}\int_{V(\tau)} \rho(e+\phi)\,dV = -\int_{A(\tau)} \rho(e+\phi)\,\vec{n}.(\vec{v}-\vec{\omega})\,dA - \int_{A(\tau)} \vec{n}.\vec{q}''\,dA - \int_{A(\tau)} \vec{n}.p\overline{\overline{I}}.\vec{v}\,dA + \int_{A(\tau)} \vec{n}.\overline{\overline{\sigma}}.\vec{v}\,dA + \int_{V(\tau)} q'''\,dV$$

- FIX CONTROL VOLUME.
- STEADY STATE.
- KINETIC ENERGY NEGLIGIBLE.
- POTENTIAL ENERGY NEGLIGIBLE.
- VISCOUS ENERGY NEGLIGIBLE.
- NO INTERNAL SOURCES.

$$-\int_A \rho u(\vec{n}.\vec{v})\,dA - \int_A \vec{n}.p\overline{\overline{I}}.\vec{v}\,dA - \int_A \vec{n}.\vec{q}''\,dA = 0$$

$$u = h - \frac{p}{\rho}$$

u : INTERNAL ENERGY

$$\int_A \vec{n}.\rho h\vec{v}\,dA + \int_A \vec{n}.\vec{q}''\,dA = 0$$

ENTHALPY EQUATION

APPLICATION TO THE CONTROL VOLUME abcd

$$E_{ab} + E_{cd} + E_{bc} + \int_{A_{da}} \vec{n}_{da}.\vec{q}''\,dA = 0$$

ENERGY CONSERVATION EQUATION - INTEGRAL FORMULATION

$$E_{ab} + E_{cd} + E_{bc} + \int_{A_{da}} \vec{n}_{da} . \vec{q}\,'' dA = 0$$

$$E_{ab} = \int_{A_{ab}} \vec{n}_{ab} \rho (u\vec{i}) h dA = -\left[\int_0^l \rho u h dy\right]_x$$

$$E_{cd} = \int_{A_{cd}} \vec{n}_{cd} \rho (u\vec{i}) h dA = \left[\int_0^l \rho u h dy\right]_{x+dx}$$

$$= \left[\int_0^l \rho u h dy\right]_x + \frac{d}{dx}\left[\int_0^l \rho u h dy\right]_x dx$$

$$E_{bc} = -\frac{d}{dx}\left[\int_0^l \rho u h_f dy\right]_x dx$$

$$\int_{A_{da}} \vec{n}_{da} . \vec{q}\,'' dA = \vec{n}_{da} . \vec{q}\,'' dx$$

$$= \vec{n}_{da} . \left[-k_f \left.\frac{dt}{dy}\right|_{y=0} \vec{j} \right] dx = k_f \left.\frac{dt}{dy}\right|_{y=0} dx$$

$$\frac{d}{dx}\left[\int_0^l \rho u h dy\right] - \frac{d}{dx}\left[\int_0^l \rho u h_f dy\right] + k_f \left.\frac{dt}{dy}\right|_{y=0} = 0$$

or

$$\frac{d}{dx}\int_0^l \rho u (h_f - h) dy = k_f \left.\frac{dt}{dy}\right|_{y=0}$$

ENERGY CONSERVATION EQUATION - INTEGRAL FORMULATION

$$\frac{d}{dx}\int_0^l \rho u(h_f - h)dy = k_f \left.\frac{dt}{dy}\right|_{y=0}$$

$$h_f - h = c_p\left(t_f - t\right)$$

$$\frac{d}{dx}\int_0^{\delta_t(x)} c_p \rho u(t_f - t)dy = k_f \left.\frac{dt}{dy}\right|_{y=0}$$

$$l \rightarrow \delta_t(x)$$

THIS IS THE INTEGRAL ENERGY EQUATION OF A STEADY, LAMINAR AND INCOMPRESSIBLE BOUNDARY LAYER.

TURBULENT BOUNDARY LAYER

- WE HAVE JUST DISCUSSED THE LAMINAR BOUNDARY LAYER.
- HOWEVER, IN MANY ENGINEERING APPLICATIONS, THE BOUNDARY LAYER IS TURBULENT.
- IN LAMINAR BOUNDARIES, MOMENTUM AND HEAT ARE TRANSPORTED ACROSS THE FLUID LAYERS ONLY BY MOLECULAR DIFFUSION.
- CONSEQUENTLY, THE CROSS FLOW OF PROPERTIES IS SMALL.
- IN TURBULENT FLOWS, THE MIXING BETWEEN ADJACENT FLUID LAYERS IS SIMULTANEOUSLY GOVERNED BY TWO MECHANISMS:
  - MOLECULAR TRANSPORT, AND
  - MACROSCOPIC TRANSPORT DUE TO FLUID LUMPS (PARTICLES).
- BECAUSE OF THE SECOND MECHANISM, MOMENTUM AND ENERGY TRANSPORT IS GREATLY ENHANCED.
- TO DISCUSS THE BASIC FEATURES OF TURBULENT BOUNDARY LAYERS WE WILL ASSUME THAT THE GOVERNING EQUATIONS CAN BE OBTAINED FROM SIMPLIFIED LAMINAR BOUNDARY LAYER EQUATIONS.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\alpha\frac{\partial^2 t}{\partial y^2}$$

LAMINAR FLOW EQUATION OVER A FLAT PLATE

$$\alpha=\frac{k_f}{\rho c_p}$$

$U(x)=const.$
i.e.,
$p=const.$

$$u=\bar{u}+u'$$
$$v=\bar{v}+v'$$
$$t=\bar{t}+t'$$

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0 \quad *$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\nu\frac{\partial^2 \bar{u}}{\partial y^2}-\frac{\partial}{\partial x}\overline{u'^2}-\frac{\partial}{\partial y}\overline{u'v'}$$

$$\bar{u}\frac{\partial \bar{t}}{\partial x}+\bar{v}\frac{\partial \bar{t}}{\partial y}=\alpha\frac{\partial^2 \bar{t}}{\partial y^2}-\frac{\partial}{\partial x}\overline{u't'}-\frac{\partial}{\partial y}\overline{v't'}$$

* SEE THE DERIVATION

TURBULENT FLOW EQUATIONS OVER A FLAT PLATE.

* DERIVATION OF THE TIME AVERAGED MASS CONSERVATION EQUATION

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$u = \bar{u} + u'$
$v = \bar{v} + v'$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0$$

TIME AVERAGE

$$\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \frac{\partial \bar{u}}{\partial x} d\tau + \frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \frac{\partial u'}{\partial x} d\tau + \frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \frac{\partial \bar{v}}{\partial y} d\tau + \frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \frac{\partial v'}{\partial y} d\tau = 0$$

or

$$\frac{\partial}{\partial x}\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \bar{u}\, d\tau + \frac{\partial}{\partial x}\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} u'\, d\tau + \frac{\partial}{\partial y}\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \bar{v}\, d\tau + \frac{\partial}{\partial y}\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} v'\, d\tau = 0$$

$$\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \bar{f}\, d\tau = \bar{f}$$

$$\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} f'\, d\tau = 0$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

TURBULENT FLOW - MOMENTUM AND ENERGY EQUATIONS

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\nu\frac{\partial^2 \bar{u}}{\partial y^2}-\frac{\partial}{\partial x}\overline{u'^2}-\frac{\partial}{\partial y}\overline{u'v'}$$

$$\bar{u}\frac{\partial \bar{t}}{\partial x}+\bar{v}\frac{\partial \bar{t}}{\partial y}=\alpha\frac{\partial^2 \bar{t}}{\partial y^2}-\frac{\partial}{\partial x}\overline{u't'}-\frac{\partial}{\partial y}\overline{v't'}$$

► NEGLECT:

$\frac{\partial}{\partial x}\overline{u'^2}$ and $\frac{\partial}{\partial x}\overline{u't'}$

$$\tau_l=\mu\frac{\partial \bar{u}}{\partial y}$$

$$q_l''=-k_f\frac{\partial \bar{t}}{\partial y}$$

$$\alpha=\frac{k_f}{\rho c_p}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial y}\mu\frac{\partial \bar{u}}{\partial y}-\frac{\partial}{\partial y}\overline{u'v'}=\frac{1}{\rho}\frac{\partial}{\partial y}\tau_l\boxed{-\frac{\partial}{\partial y}\overline{u'v'}}$$

$$\bar{u}\frac{\partial \bar{t}}{\partial x}+\bar{v}\frac{\partial \bar{t}}{\partial y}=\alpha\frac{\partial^2 \bar{t}}{\partial y^2}-\frac{\partial}{\partial y}\overline{v't'}=-\frac{1}{\rho c_p}\frac{\partial}{\partial y}q_l''\boxed{-\frac{\partial}{\partial y}\overline{v't'}}$$

$\tau_l=\mu\frac{\partial \bar{u}}{\partial y}$: REPRESENTS THE SHEAR STRESS DUE TO MOLECULAR TRANSPORT OF MOMENTUM.

$q_l''=-k_f\frac{\partial \bar{t}}{\partial y}$ REPRESENTS THE HEAT FLUX DUE TO MOLECULAR TRANSPORT OF HEAT.

- TO UNDERSTAND THE MEANING OF:

$$\frac{\partial}{\partial y}\overline{u'v'} \quad and \quad \frac{\partial}{\partial y}\overline{v't'}$$

CONSIDER THE TWO DIMENSIONAL FLOW IN WHICH THE MEAN VALUE OF THE VELOCITY IS PARALLEL TO THE x-DIRECTION.

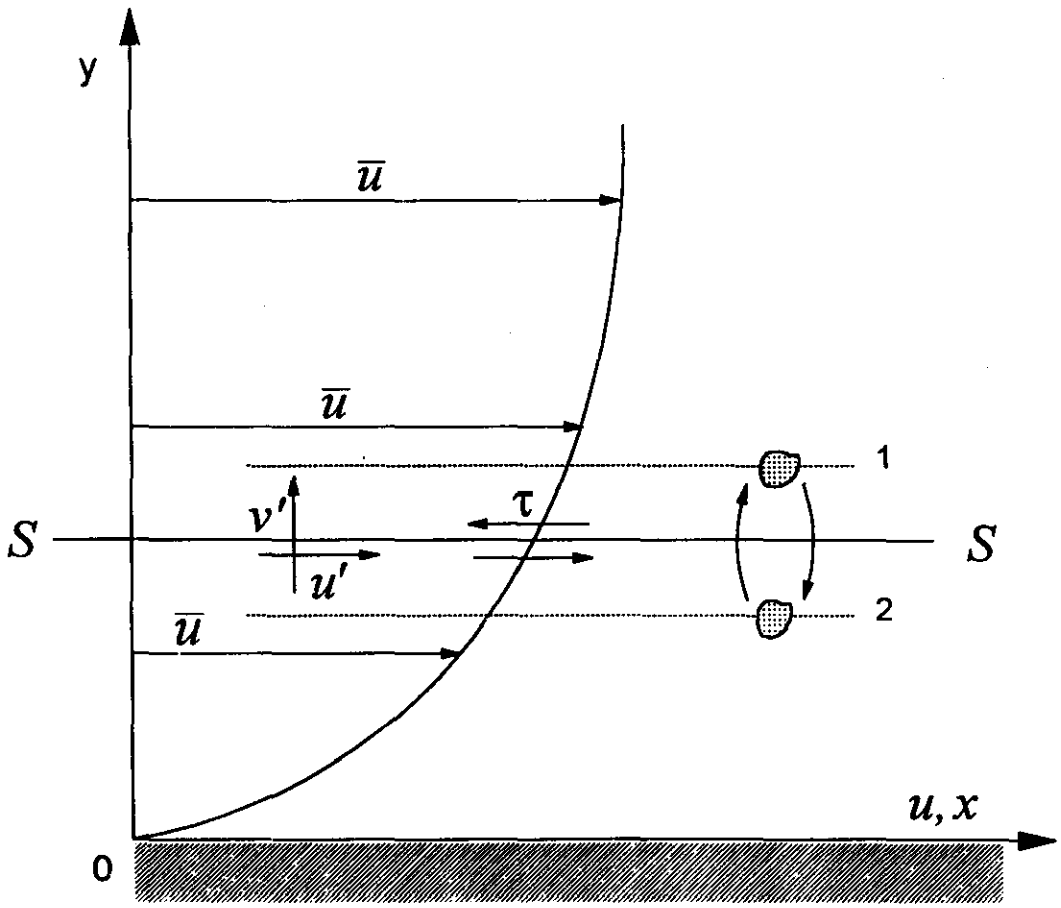

Figure 4.11 Turbulent momentum exchange in two dimensional flow.

- BECAUSE OF THE TURBULENT NATURE OF THE FLOW, THE INSTANTANEOUS VELOCITY OF THE FLUID CHANGES CONTINUOUSLY:
  - IN DIRECTION, AND
  - IN MAGNITUDE.

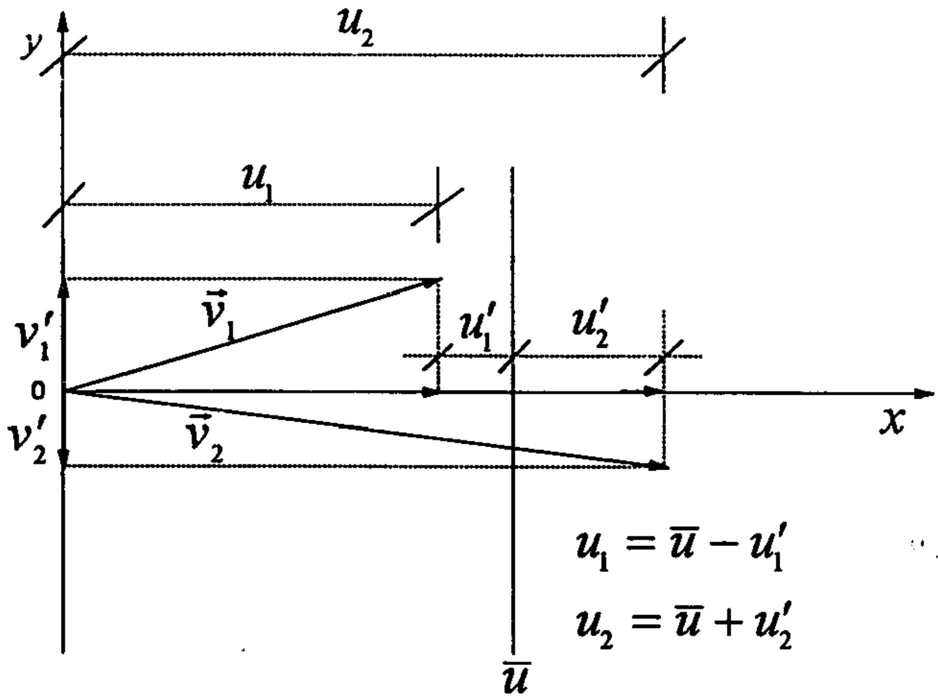

Figure 4.12 Instantaneous turbulent velocities

- THE INSTANTANEOUS VELOCITY COMPONENTS ARE:

$$u = \overline{u} + u'$$

$$v = v'$$

- WHILE DISCUSSING THE VISCOSITY, WE HAVE SEEN THAT:

  - AN EXCHANGE OF MOLECULES BETWEEN THE FLUID LAYERS ON EITHER SIDE OF THE PLANE SS PRODUCES A CHANGE IN THE x-DIRECTION MOMENTUM.
  - THIS CHANGE IS CAUSED BY THE EXISTENCE OF A GRADIENT IN THE x-DIRECTION VELOCITY.
  - THE MOMENTUM CHANGE PRODUCES A SHEARING FORCE IN THE FLUID PARALLEL TO x-DIRECTION AND DENOTED BY τ_l.

- IF TURBULENT FLOW VELOCITY FLUCTUATIONS OCCUR BOTH IN x- AND y-DIRECTIONS (CASE STUDIED):

  - THE y-DIRECTION FLUCTUATIONS, v', TRANSPORT FLUID LUMPS (LARGER THAN THE MOLECULAR TRANSPORT).
  - INSTANTANEOUS RATE OF MASS TRANSPORT PER UNIT AREA AND PER UNIT TIME ACROSS SS IS:

$$\rho v'$$

  - INSTANTANEOUS RATE OF TRANSFER IN THE y-DIRECTION OF x-DIRECTION MOMENTUM PER UNIT AREA AND TIME ACROSS SS IS:

$$-\rho v'(\overline{u} + u')$$

THE MEANING OF THE "MINUS" SIGN WILL BE DISCUSSED LATER.

- THE TIME AVERAGE OF THE x-DIRECTION MOMENTUM TRANSFER CREATES A TURBULENT SHEAR STRESS OR REYNOLDS STRESS, τ_t :

$$\tau_t = -\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \rho v'(\overline{u} + u')\,d\tau$$

$$\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} (\rho v')\overline{u}\,d\tau = 0$$

$$\tau_t = -\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} (\rho v')u'\,d\tau = -\overline{(\rho v')u'}$$

$$\rho = const.$$

$$\tau_t = -\rho\overline{v'u'}$$

- $\overline{v'u'}$ IS THE TIME AVERAGE OF THE PRODUCT OF u' AND v' ; IT IS DIFFERENT FROM ZERO.

- TO UNDERSTAND THE REASON FOR THE MINUS SIGN CONSIDER THE FOLLOWING FIGURE:

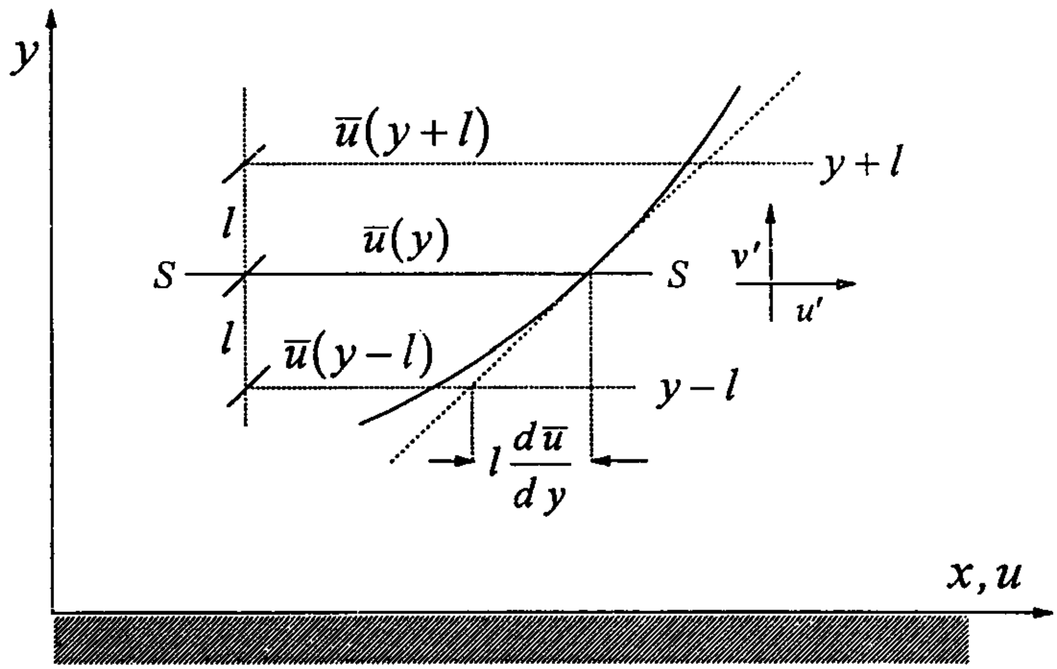

Figure 4.13 Mixing length for momentum transfer in turbulent flow

- THE FLUID LUMPS WHICH TRAVEL UPWARD ($v' > 0$) ARRIVE IN A LAYER IN THE FLUID WHERE THE MEAN VELOCITY $\overline{u}$ IS LARGER THAN THE VELOCITY OF THE LAYER FROM WHICH THEY COME.
- WE WILL ASSUME THAT THESE LUMPS KEEP THEIR ORIGINAL VELOCITY $\overline{u}$ DURING THEIR MIGRATION.
- THEY WILL, THEREFORE, TEND TO SLOW DOWN THE FLUID LUMPS EXISTING IN THEIR DESTINATION LAYER.
- THEREBY, THEY WILL GIVE RISE TO A NEGATIVE u' .
- CONVERSELY IF v' IS NEGATIVE.
- THE OBSERVED VALUE OF u' AT THE NEW DESTINATION WILL BE POSITIVE .

- CONSEQUENTLY, ON THE AVERAGE:
  - A POSITIVE v' IS ASSOCIATED WITH A NEGATIVE u', AND
  - VICE VERSA.
- THE TIME AVERAGE OF $\overline{v'u'}$ IS NOT ZERO BUT A NEGATIVE QUANTITY.

$$\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial y}\tau_l-\frac{\partial}{\partial y}\overline{u'v'}$$

$$\tau_t=-\rho\overline{v'u'}$$

$$\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial y}(\tau_l+\tau_t)$$

$\tau=\tau_l+\tau_t$ IS CALLED TOTAL SHEAR STRESS IN TURBULENT FLOW.

- THE TURBULENT MOMENTUM TRANSPORT CAN BE RELATED TO THE TIME-AVERAGE VELOCITY GRADIENT:

$$\frac{\partial \bar{u}}{\partial y}$$

BY USING THE "MEAN FREE PATH" CONCEPT INTRODUCED DURING THE STUDY OF THE MOLECULAR MOMENTUM TRANSPORT.

  - IN TURBULENT FLOWS, THE DISTANCE $''l''$ TRAVELED BY THE FLUID LUMPS IN THE DIRECTION NORMAL TO THE MEAN FLOW WHILE MAINTAINING THEIR IDENTITY AND PHYSICAL PROPERTIES IS CALLED "**MIXING LENGTH.**"
  - CONSIDER A FLUID LUMP LOCATED AT A DISTANCE $''l''$ ABOVE AND BELOW THE SURFACE SS.

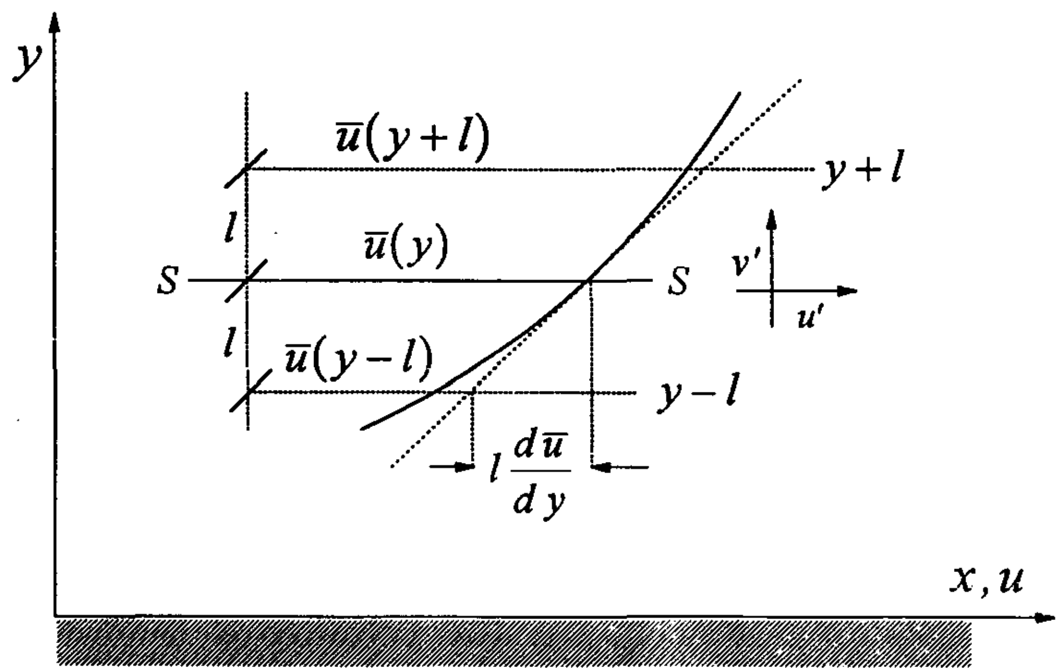

Figure 4.13 Mixing length for momentum transfer in turbulent flow.

- AFTER DEVELOPING IN TAYLOR SERIES, THE VELOCITY OF A LUMP AT $(y+l)$ IS:

$$\overline{u}(y+l) \cong \overline{u}(y) + l\frac{\partial \overline{u}}{\partial y}$$

WHEREAS AT $(y-l)$

$$\overline{u}(y-l) \cong \overline{u}(y) - l\frac{\partial \overline{u}}{\partial y}$$

- IF THE FLUID LUMP MOVES FROM LAYER $(y-l)$ TO THE LAYER y UNDER THE INFLUENCE OF A POSITIVE v', ITS VELOCITY PARALLEL TO x-DIRECTION WILL BE SMALLER THAN THE VELOCITY PREVAILING IN THE LAYER y BY AN AMOUNT:

$$\overline{u}(y-l) - \overline{u}(y) \cong -l\frac{\partial \overline{u}}{\partial y}$$

- SIMILARLY, IF A LUMP OF FLUID ARRIVES TO THE LAYER y FROM LAYER $(y+l)$ UNDER THE INFLUENCE OF A NEGATIVE v' ITS VELOCITY WILL BE HIGHER BY AN AMOUNT:

$$\overline{u}(y+l) - \overline{u}(y) \cong l\frac{\partial \overline{u}}{\partial y}$$

- THESE DIFFERENCES IN $\overline{u}$ -VELOCITIES CONSTITUTE THE BASIS OF u' FLUCTUATIONS:

$$u' \cong l\frac{\partial \overline{u}}{\partial y}$$

- TURBULENT SHEAR STRESS.

$$\tau_t = -\rho \overline{v'u'}$$

$$u' \cong l \frac{\partial \overline{u}}{\partial y}$$

$$\tau_t = -\rho \overline{v'l} \frac{\partial \overline{u}}{\partial y}$$

DEFINING "APPARENT KINEMATIC VISCOSITY" AS: $\varepsilon_m = -\overline{v'l}$

$$\tau_t = \rho \varepsilon_m \frac{\partial \overline{u}}{\partial y}$$

- TOTAL SHEAR STRESS:

$$\tau = \tau_l + \tau_t$$

$$\tau_l = \mu \frac{\partial \overline{u}}{\partial y}$$

$$\tau_t = \rho \varepsilon_m \frac{\partial \overline{u}}{\partial y}$$

$$\tau = \tau_l + \tau_t = \rho(\nu + \varepsilon_m) \frac{\partial \overline{u}}{\partial y}$$

$$\nu = \frac{\mu}{\rho}$$

- USUALLY v' IS OF THE SAME ORDER AS u'.
- ε_m IS NOT A PHYSICAL PROPERTY AS ν.
- ε_m DEPENDS ON THE MOTION OF THE FLUID, Re-NUMBER, etc.
- ε_m VARIES FROM POINT TO POINT IN THE FLOW; IT VANISHES NEAR THE WALL.
- ε_m / ν CAN GO AS HIGH AS 500.
- ν CAN, THEREFORE BE IGNORED IN COMPARISON WITH ε_m.

ε_m : APPARENT KINEMATIC VISCOSITY

TRANSFER OF ENERGY

- THE TRANSFER OF ENERGY IN A TURBULENT FLOW CAN BE MODELED IN A WAY SIMILAR TO THAT OF THE MOMENTUM.

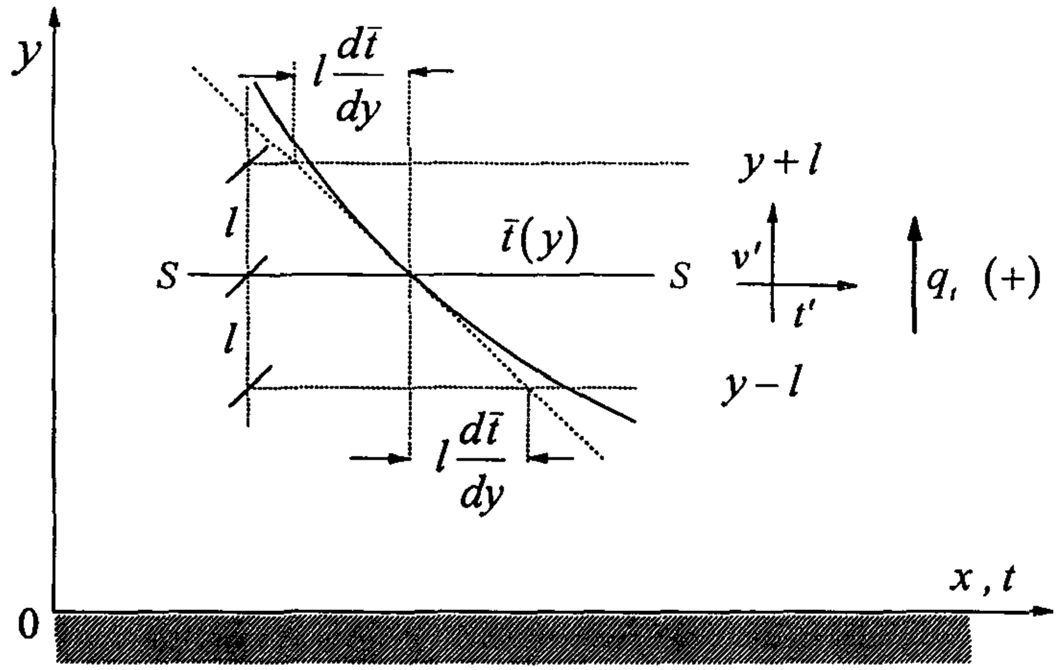

Figure 4.14 Mixing length for energy transfer in turbulent flow.

► INSTANTANEOUS ENERGY TRANSPORT PER UNIT AREA AND UNIT TIME IN THE y-DIRECTION:

$$\rho c_p v'(t)$$

WHERE:

$$t = \bar{t} + t'$$

i.e.,

$$\rho c_p v'(\bar{t} + t')$$

► THE TIME AVERAGE OF TURBULENT HEAT TRANSFER:

$$q_t'' = \frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \rho c_p v'(\bar{t} + t')\,d\tau$$

$$\frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \left(\rho c_p v'\right)\bar{t}\,d\tau = 0$$

$$q_t'' = \frac{1}{\Delta\tau_o}\int_{\Delta\tau_o} \left(\rho c_p v'\right)t'\,d\tau = \overline{\left(\rho c_p v'\right)t'}$$

$$\rho = const.$$
$$c_p = const.$$

$$q_t'' = \rho c_p \overline{v't'}$$

► ENERGY EQUATION FOR A TURBULENT FLOW:

$$\bar{u}\frac{\partial \bar{t}}{\partial x} + \bar{v}\frac{\partial \bar{t}}{\partial y} = \alpha\frac{\partial^2 \bar{t}}{\partial y^2} - \frac{\partial}{\partial y}\overline{v't'} = -\frac{1}{\rho c_p}\frac{\partial}{\partial y}q_l'' - \frac{\partial}{\partial y}\overline{v't'}$$

$$q_t'' = \rho c_p \overline{v't'}$$

$$\bar{u}\frac{\partial \bar{t}}{\partial x} + \bar{v}\frac{\partial \bar{t}}{\partial y} = -\frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(q_l'' + q_t''\right)$$

$q'' = q_l'' + q_t''$: TOTAL HEAT FLUX IN A TURBULENT FLOW.

- USING THE CONCEPT OF MIXING LENGTH WE CAN WRITE THAT:

$$t' \cong l \frac{\partial \bar{t}}{\partial y}$$

$$q_t'' = \rho c_p \overline{v't'}$$

$$t' \cong l \frac{\partial \bar{t}}{\partial y}$$

$$q_t'' = \rho c_p \overline{v't'} = -\rho c_p \overline{v'l} \frac{\partial \bar{t}}{\partial y}$$

- $v't'$ IS POSITIVE IN THE AVERAGE.
- THE MINUS SIGN IS INTRODUCED TO RESPECT THE CONVENTION THAT HEAT FLOW IS POSITIVE IN THE DIRECTION OF y POSITIVE.
- THEREFORE, THE SECOND LAW OF THERMODYNAMICS IS SATISFIED.
- TURBULENT HEAT TRANSFER IS THEN WRITTEN AS:

$$q_t'' = -\rho c_p \overline{v'l} \frac{\partial \bar{t}}{\partial y}$$

$$\varepsilon_h = \overline{v'l}$$

$$q_t'' = -\rho c_p \varepsilon_h \frac{\partial \bar{t}}{\partial y}$$

- TOTAL HEAT TRANSFER

$$q'' = q_l'' + q_t''$$

$$q_l'' = -k_f \frac{\partial \bar{t}}{\partial y}$$

$$q_t'' = -\rho c_p \varepsilon_h \frac{\partial \bar{t}}{\partial y}$$

$$q'' = -\left(k_f + c_p \rho \varepsilon_h\right) \frac{\partial \bar{t}}{\partial y}$$

or

$$q'' = -c_p \rho (\alpha + \varepsilon_h) \frac{\partial \bar{t}}{\partial y}$$

$\alpha = \dfrac{k_f}{c_p \rho}$: MOLECULAR DIFFUSIVITY OF HEAT

ε_h : EDDY DIFFUSIVITY OF HEAT

FORCED CONVECTION OVER A FLAT PLATE

OBJECTIVES: DETERMINE THE WALL FRICTION AND HEAT TRANSFER COEFFICIENTS IN LAMINAR AND TURBULENT BOUNDARY LAYERS.

IN ORDER TO REACH RAPIDLY THE OBJECTIVES **"INTEGRAL MOMENTUM AND ENERGY CONSERVATION EQUATIONS"** WILL BE USED.

LAMINAR BOUNDARY LAYER

- IN LAMINAR BOUNDARY LAYER, THE FLUID MOTION IS VERY ORDERLY.
- THE FLUID MOTION ALONG A STREAMLINE HAS VELOCITY COMPONENTS IN x AND y DIRECTIONS (u AND v).
- THE VELOCITY COMPONENT v, NORMAL TO THE WALL, CONTRIBUTES SIGNIFICANTLY TO MOMENTUM AND ENERGY TRANSFER THROUGH THE BOUNDARY.
- FLUID MOTION NORMAL TO THE WALL IS BROUGHT ABOUT BY THE BOUNDARY LAYER GROWTH IN THE x-DIRECTION.

LAMINAR BOUNDARY LAYER.

- CONSIDER A FLAT PLATE OF CONSTANT TEMPERATURE PLACED PARALLEL TO THE INCIDENT FLOW AS ILLUSTRATED IN THE FOLLOWING FIGURE.

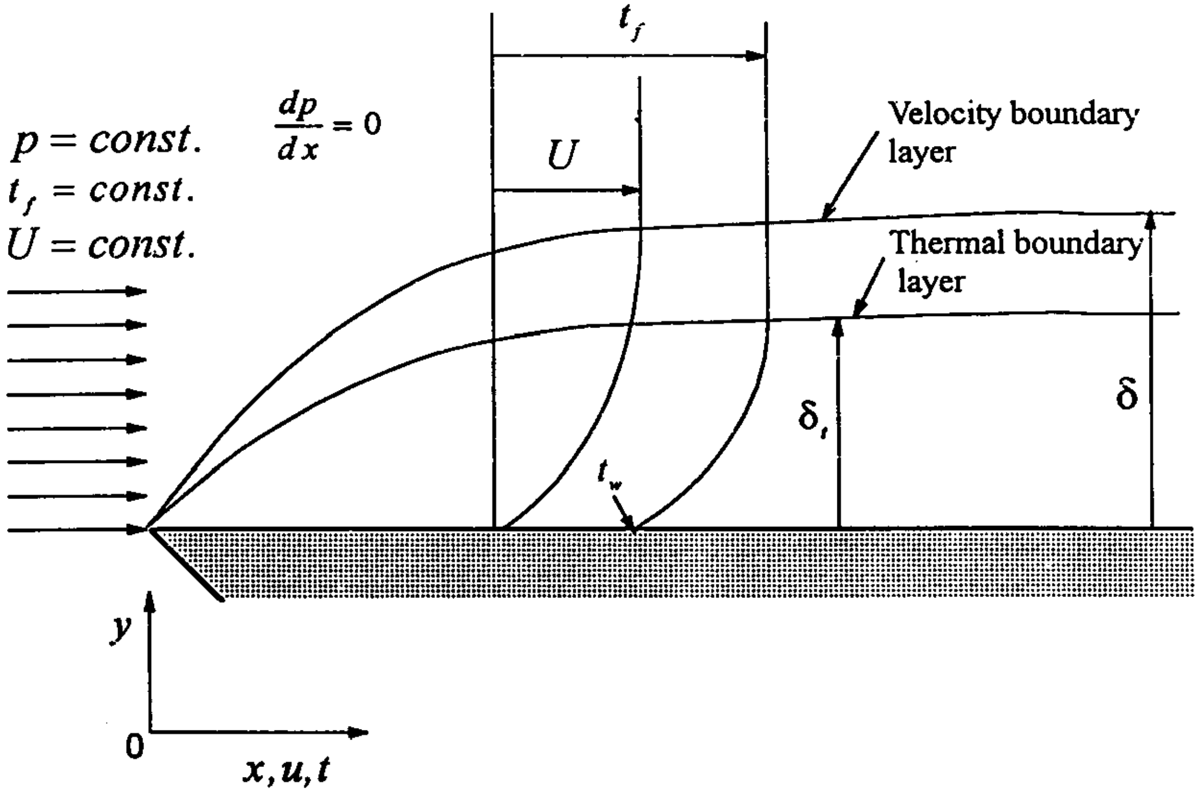

Figure 4.15 Velocity and thermal boundary layers for a laminar flow past a flat plate.

- $U(x) = const. = U$
- $p(x) = const. = p$
- PHYSICAL PROPERTIES ARE CONSTANT.

LAMINAR BOUNDARY LAYER.

VELOCITY BOUNDARY LAYER- BOUNDARY LAYER THICKNESS.

$$\rho \frac{d}{dx}\int_0^{\delta(x)} (U(x)-u)u\,dy + \rho \frac{dU(x)}{dx}\int_0^{\delta(x)} (U(x)-u)\,dy = \tau_w$$

$$U(x) = const. = U$$

$$p(x) = const. = p$$

PHYSICAL PROPERTIES ARE CONSTANT.

$$\rho \frac{d}{dx}\int_0^{\delta(x)} (U - u)u\,dy = \tau_w$$

$$u(x,y) = a(x) + b(x)y + c(x)y^2 + d(x)y^3$$

BOUNDARY CONDITIONS

$y = 0 \quad u = 0$

$y = \delta \quad u = U$

$y = \delta \quad \frac{\partial u}{\partial y} = 0$

$y = 0 \quad \frac{\partial^2 u}{\partial y^2} = 0$

$$a = 0,\ b = \frac{3}{2}\frac{U}{\delta},\ c = 0,\ d = -\frac{1}{2}\frac{U}{\delta^3}$$

LAMINAR BOUNDARY LAYER - VELOCITY BOUNDARY LAYER

$$\rho \frac{d}{dx}\int_0^{\delta(x)} (U(x)-u)u\,dy = \tau_w$$

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{3}{2}\mu\frac{U}{\delta}$$

INTEGRATION

$$\frac{39}{280}\rho U^2 \frac{d\delta}{dx} = \frac{3}{2}\mu\frac{U}{\delta}$$

or

$$\delta\, d\delta = \frac{140}{13}\frac{\mu}{\rho}\frac{1}{U}dx$$

INTEGRATION

$$\delta = 4.64\sqrt{\frac{\mu}{\rho}\frac{1}{U}x + const.}$$

$x = 0 \quad \delta = 0$

i.e., $const. = 0$

$$\delta = 4.64\sqrt{\frac{\mu}{\rho}\frac{1}{U}x} \quad \text{or} \quad \frac{\delta}{x} = \frac{4.64}{\sqrt{\frac{\rho U x}{\mu}}} = \frac{4.64}{Re_x^{1/2}}$$

LAMINAR BOUNDARY LAYER - VELOCITY BOUNDARY LAYER

VELOCITY BOUNDARY LAYER- FRICTION COEFFICIENT

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{3}{2}\mu\frac{U}{\delta}$$

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\frac{\rho U x}{\mu}}} = \frac{4.64}{Re_x^{1/2}}$$

$$\tau_w = 0.323\frac{\rho U^2}{Re_x^{1/2}}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$C_f = \frac{0.646}{\sqrt{Re_x}}$$

LOCAL FRICTION COEFFICIENT

AVERAGE FRICTION COEFFICIENT

$$C_x = \frac{\int_0^x C_f\,dx}{\int_0^x dx}$$

$$C_f = \frac{1.292}{\sqrt{Re_x}}$$

AVERAGE FRICTION COEFFICIENT

LAMINAR BOUNDARY LAYER.

THERMAL BOUNDARY LAYER- BOUNDARY LAYER THICKNESS.

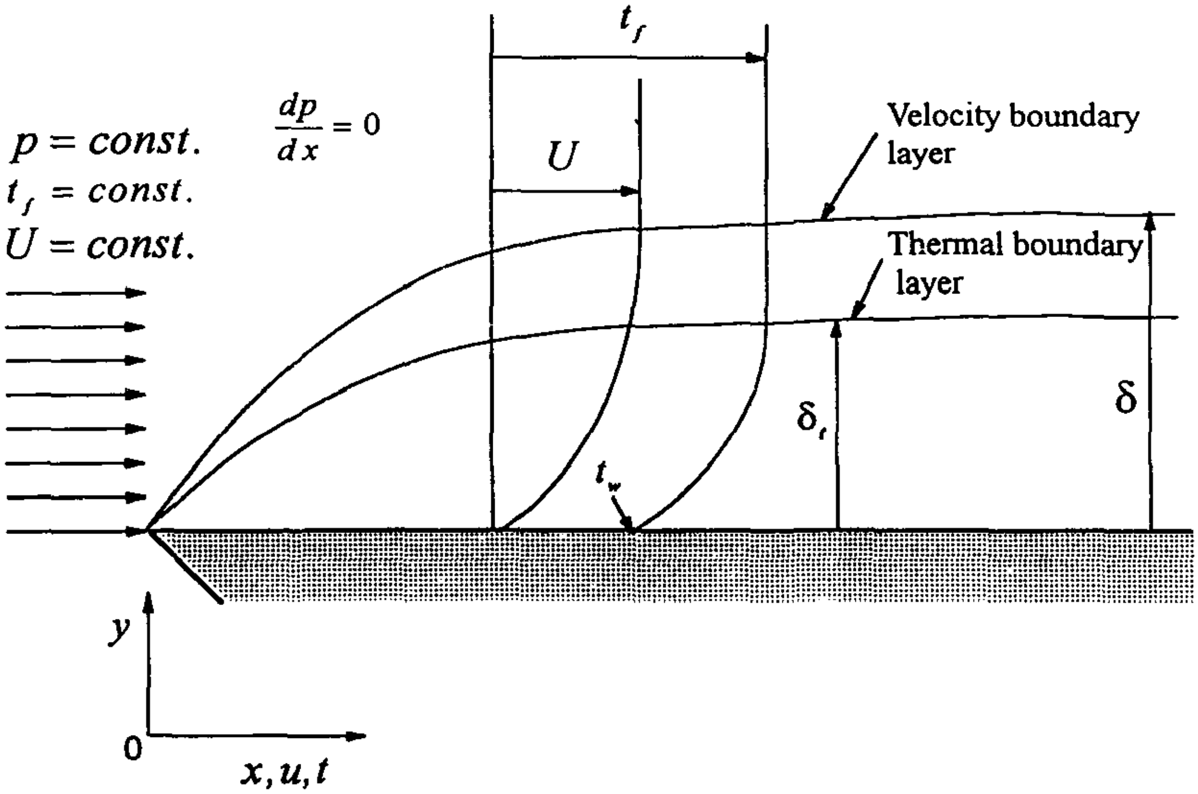

Figure 4.15 Velocity and thermal boundary layers for a laminar flow past a flat plate.

- TEMPERATURE OF THE PLATE IS CONSTANT.

LAMINAR BOUNDARY LAYER - THERMAL BOUNDARY LAYER THICKNESS

$$\frac{d}{dx}\int_0^{\delta_t(x)} c_p \rho u (t_f - t)\, dy = k_f \left.\frac{dt}{dy}\right|_{y=0}$$

$$\theta = t - t_w$$

$$\theta_w = t_f - t_w$$

$$\frac{d}{dx}\int_0^{\delta_t} c_p \rho u (\theta_w - \theta)\, dy = k_f \left.\frac{d\theta}{dy}\right|_{y=0}$$

$$t(x,y) = a(x) + b(x)y + c(x)y^2 + d(x)y^3$$

BOUNDARY CONDITIONS

$y = 0 \quad t = t_w$

$y = \delta_t \quad t = t_f$

$y = \delta_t \quad \dfrac{\partial t}{\partial y} = 0$

$y = 0 \quad \dfrac{\partial^2 t}{\partial y^2} = 0$

$$\frac{t - t_w}{t_f - t_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

$$\theta = t - t_w$$
$$\theta_w = t_f - t_w$$

$$\frac{\theta}{\theta_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

LAMINAR BOUNDARY LAYER - THERMAL BOUNDARY LAYER THICKNESS

$$\frac{d}{dx}\int_0^{\delta_t} c_p \rho u(\theta_w - \theta)\,dy = k_f \left.\frac{d\theta}{dy}\right|_{y=0}$$

$$\frac{\theta}{\theta_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\theta_w U \frac{d}{dx}\int_0^{\delta_t}\left[1 - \frac{3}{2}\frac{y}{\delta_t} + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy = \frac{3}{2}\alpha\frac{\theta_w}{\delta_t}$$

$$\xi = \frac{\delta_t}{\delta}$$

$$\alpha = \frac{k_f}{\rho c_p}$$

ALGEBRAIC MANIPULATIONS AND INTEGRATION

$$\frac{d}{dx}\left[\delta\left(\frac{3}{20}\xi^2 - \frac{3}{280}\xi^4\right)\right] = \frac{3}{2}\frac{\alpha}{\xi\delta}\frac{1}{U}$$

LAMINAR BOUNDARY LAYER - THERMAL BOUNDARY LAYER THICKNESS

$$\frac{d}{dx}\left[\delta\left(\frac{3}{20}\xi^2 - \frac{3}{280}\xi^4\right)\right] = \frac{3}{2}\frac{\alpha}{\xi\delta}\frac{1}{U}$$

If $\xi \le 1$

$$\frac{3}{20}\xi^2 > \frac{3}{280}\xi^4$$

IGNORE $\frac{3}{280}\xi^4$

$$\frac{3}{20}\frac{d}{dx}\delta\xi^2 = \frac{3}{2}\frac{\alpha}{\xi\delta}\frac{1}{U}$$

or

$$\frac{1}{10}\left(\xi^3\delta\frac{d\delta}{dx} + 2\xi^2\delta^2\frac{d\xi}{dx}\right) = \frac{\alpha}{U}$$

$$\delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\mu}{\rho}\frac{1}{U}$$

$$\delta = 4.64\sqrt{\frac{\mu}{\rho}\frac{x}{U}}$$

$$\frac{14}{13}\frac{\mu}{\rho}\frac{1}{\alpha}\left(\xi^3 + 4x\xi^2\frac{d\xi}{dx}\right) = 1$$

LAMINAR BOUNDARY LAYER - THERMAL BOUNDARY LAYER THICKNESS

$$\frac{14}{13}\frac{\mu}{\rho}\frac{1}{\alpha}\left(\xi^3 + 4x\xi^2\frac{d\xi}{dx}\right) = 1$$

$$Pr = \frac{c_p\mu}{k_f} = \frac{\mu}{\rho}\frac{\rho c_p}{k_f} = \frac{\mu}{\rho}\frac{1}{\alpha}$$

$$\left(\xi^3 + 4x\xi^2\frac{d\xi}{dx}\right) = \frac{13}{14}\frac{1}{Pr}$$

or

$$\left(\xi^3 + \frac{4}{3}x\frac{d}{dx}\xi^3\right) = \frac{13}{14}\frac{1}{Pr}$$

$$y = \xi^3$$

$$y + \frac{4}{3}x\frac{dy}{dx} = \frac{13}{14}\frac{1}{Pr}$$

INTEGRATION

$$y = \frac{13}{14}\frac{1}{Pr} + Cx^{-\frac{3}{4}}$$

or

$$\xi^3 = \frac{13}{14}\frac{1}{Pr} + Cx^{-\frac{3}{4}}$$

LAMINAR BOUNDARY LAYER - THERMAL BOUNDARY LAYER THICKNESS

$$\xi^3 = \frac{13}{14}\frac{1}{Pr} + Cx^{-\frac{3}{4}}$$

SINCE THE PLATE IS HEATED FROM THE LEADING EDGE, *C* MUST BE ZERO TO AVOID INDETERMINATE SOLUTION AT THE LEADING EDGE.

$$\xi^3 = \frac{13}{14}\frac{1}{Pr}$$

THIS IS THE VARIATION OF THE THERMAL BOUNDARY LAYER THICKNESS

$$\xi = \frac{\delta_t}{\delta} = \frac{1}{1.026\,Pr^{1/3}}$$

- WE ASSUMED THAT : $\xi \leq 1$
- THIS ASSUMPTION IS VALID FOR: $Pr \geq 0.7$
- MOST OF THE GASES AND LIQUIDS HAVE *Pr* - NUMBERS HIGHER THAN 0.7.
- LIQUID METALS CONSTITUTE AN EXCEPTION; THEIR *Pr* - NUMBERS ARE IN THE ORDER OF MAGNITUDE OF 0.01.
- CONSEQUENTLY, THE ABOVE ANALYSIS CANNOT BE APPLIED TO LIQUID METALS.

LAMINAR BOUNDARY LAYER -LOCAL HEAT TRANSFER COEFFICIENT.

THERMAL BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

$$h_{cx} = \frac{-k_f \left(\frac{\partial t}{\partial y} \right)_{y=0}}{t_w - t_f}$$

$$\theta = t - t_w$$

$$\theta_w = t_f - t_w$$

$$h_{cx} = \frac{k_f \left(\frac{\partial \theta}{\partial y} \right)_{y=0}}{\theta_w}$$

$$\frac{\theta}{\theta_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t} \right)^3$$

$$\left(\frac{\partial \theta}{\partial y} \right)_{y=0}$$

$$\left(\frac{\partial \theta}{\partial y} \right)_{y=0} = \frac{3}{2}\frac{\theta_w}{\delta_t} = \frac{3}{2}\frac{\theta_w}{\xi \delta}$$

$$h_{cx} = \frac{3}{2}\frac{k_f}{\xi \delta}$$

FORCED CONVECTION OVER A FLAT PLATE

LAMINAR BOUNDARY LAYER -LOCAL HEAT TRANSFER COEFFICIENT.

$$h_{cx} = \frac{3}{2}\frac{k_f}{\xi\delta}$$

$$\xi = \frac{\delta_t}{\delta} = \frac{1}{1.026\,Pr^{1/3}}$$

$$\delta = 4.64\sqrt{\frac{\mu}{\rho}\frac{x}{U}}$$

$$h_{cx} = 0.332 k_f \sqrt[3]{Pr}\sqrt{\frac{\rho U}{\mu x}}$$

$$\times\frac{x}{k_f}$$

$$\frac{h_{cx}x}{k_f} = 0.332\sqrt[3]{Pr}\sqrt{\frac{\rho U x}{\mu}}$$

$$Nu_x = \frac{h_{cx}x}{k_f}$$

$$Re_x = \frac{\rho U x}{\mu}$$

LOCAL HEAT TRANSFER COEFFICIENT

$$Nu_x = 0.332\sqrt[3]{Pr}\sqrt{Re_x}$$

AVERAGE COEFFICIENT

$$\overline{h}_c = \frac{\int_0^L h_{cx}\,dx}{\int_0^L dx}$$

$$Re_L = \frac{\rho U L}{\mu}$$

AVERAGE HEAT TRANSFER COEFFICIENT

$$\overline{Nu_L} = 0.664\sqrt[3]{Pr}\sqrt{Re_L}$$

LAMINAR BOUNDARY LAYER -LOCAL HEAT TRANSFER COEFFICIENT.

- IN THE ABOVE DISCUSSION, IT IS ASSUMED THAT THE FLUID PROPERTIES ARE CONSTANT.
- IF THERE IS A SUBSTANTIAL DIFFERENCE BETWEEN THE WALL AND FREE STREAM TEMPERATURES, THE FLUID PROPERTIES ARE CALCULATED AT THE "MEAN FILM TEMPERATURE."

$$t_m = \frac{t_w + t_f}{2}$$

- LOCAL AND AVERAGE CONVECTION HEAT TRANSFER COEFFICIENT DERIVED ABOVE ARE VALID FOR:

$$Pr \geq 0.7$$

$$Re_x \leq 5 \times 10^5$$

- FOR A CONSTANT SURFACE HEAT FLUX, THE CONVECTION HEAT TRANSFER COEFFICIENT IS GIVEN BY:

$$Nu_x = 0.453\, Re_x^{1/2}\, Pr^{1/3}$$

TURBULENT BOUNDARY LAYER

- A TURBULENT BOUNDARY LAYER IS CHARACTERIZED BY VELOCITY FLUCTUATIONS.
- THESE FLUCTUATIONS ENHANCE CONSIDERABLY THE MOMENTUM AND ENERGY TRANSFER, i.e., INCREASE:
  - SURFACE FRICTION, AND
  - HEAT TRANSFER COEFFICIENT.
- TURBULENT BOUNDARY LAYER DOES NOT START DEVELOPING WITH THE LEADING EDGE OF THE PLATE.

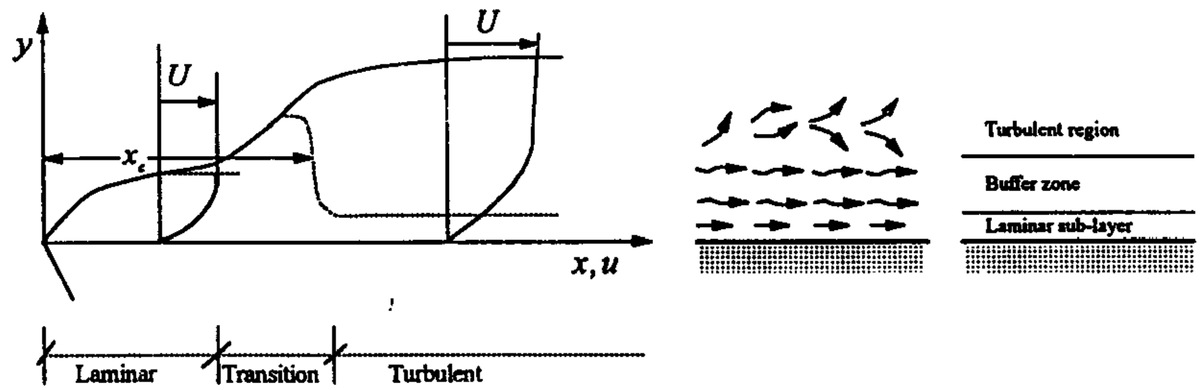

Figure 4.16 The development of laminar and turbulent layers on a flat plate

- THE BOUNDARY LAYER IS INITIALLY LAMINAR.
- AT SOME DISTANCE FROM THE LEADING EDGE, LAMINAR FLOW BECOMES UNSTABLE.
- A GRADUAL TRANSITION TO TURBULENT FLOW OCCURS.

TURBULENT BOUNDARY LAYER

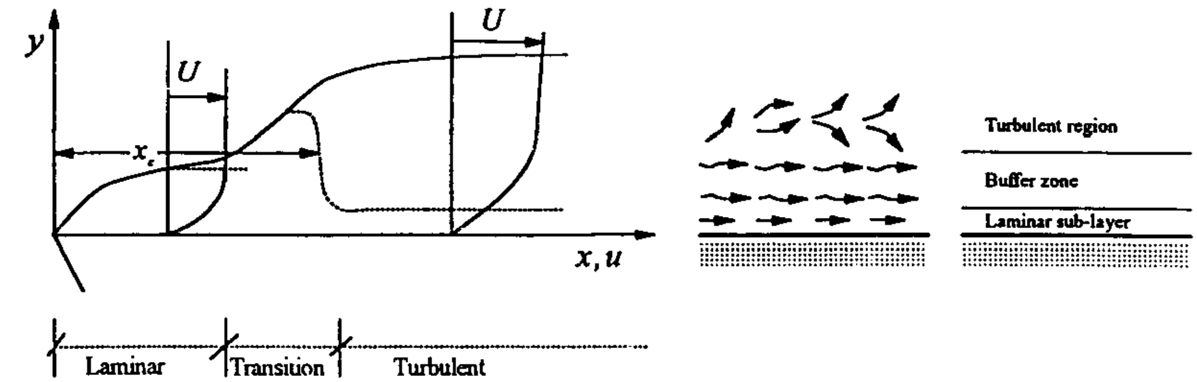

Figure 4.16 The development of laminar and turbulent layers on a flat plate

- THE TURBULENT REGION IS CHARACTERIZED BY A HIGHLY RANDOM, THREE DIMENSIONAL MOTION OF FLUID LUMPS.

- THE TRANSITION TO TURBULENCE IS ACCOMPANIED BY AN INCREASE OF:
  - THE BOUNDARY LAYER THICKNESS,
  - THE WALL SHEAR STRESS, AND
  - THE CONVECTION HEAT TRANSFER COEFFICIENT.

- IN THE TURBULENT BOUNDARY LAYER THREE REGIONS EXISTS:
  - **LAMINAR SUBLAYER** WHERE:
    - DIFFUSION DOMINATES PROPERTY TRANSPORT, AND
    - THE VELOCITY AND TEMPERATURE PROFILES ARE LINEAR.
  - **BUFFER ZONE** WHERE MOLECULAR DIFFUSION AND TURBULENT MIXING ARE COMPARABLE.
  - **TURBULENT ZONE** WHERE THE PROPERTY TRANSPORT IS DOMINATED BY TURBULENT MIXING.

TURBULENT BOUNDARY LAYER

- DESPITE THE PRESENCE OF A TRANSITION ZONE, IT IS CUSTOMARY TO ASSUME THAT THE TRANSITION FROM LAMINAR TO TURBULENT BOUNDARY LAYER OCCURS SUDDENLY.

- THE TRANSITION LOCATION x_c IS TIED TO REYNOLDS NUMBER:

$$Re_x = \frac{\rho U x}{\mu}$$

- IF $Re_x \geq 5 \times 10^5$, THE BOUNDARY LAYER IS TURBULENT.

- ANALYTICAL STUDY OF THE TURBULENT BOUNDARY LAYER IS COMPLEX:
  - THIS IS DUE TO THE FACT THAT ε_m IS NOT A PROPERTY OF THE FLUID.

- HERE, BY USING A SIMPLE APPROACH, WE WILL DISCUSS FOR A TURBULENT BOUNDARY LAYER:

  1. THE THICKNESS,
  2. THE FRICTION COEFFICIENT, AND
  3. THE HEAT TRANSFER COEFFICIENT.

TURBULENT BOUNDARY LAYER

VELOCITY BOUNDARY LAYER - BOUNDARY LAYER THICKNESS

- THE GENERAL CHARACTERISTICS OF A TURBULENT BOUNDARY LAYER RESEMBLE TO THOSE OF THE LAMINAR BOUNDARY LAYER.

- THE TIME AVERAGE VELOCITY VARIES RAPIDLY FROM ZERO AT THE WALL TO THE UNIFORM VALUE OF THE POTENTIAL CORE.

- BECAUSE OF THE TRANSVERSE FLUCTUATIONS, THE VELOCITY DISTRIBUTION IS MUCH MORE CURVED NEAR THE WALL THAN THAT IN THE LAMINAR FLOW.

- HOWEVER, THIS DISTRIBUTION IS MORE UNIFORM AT THE OUTER EDGE OF THE BOUNDARY LAYER THAN THE LAMINAR COUNTERPART.

- EXPERIMENTS HAVE SHOWN THAT THE VELOCITY DISTRIBUTION IN A TURBULENT BOUNDARY LAYER CAN BE ADEQUATELY DESCRIBED BY:

$$\frac{\overline{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

ONE SEVENTH LAW

- THIS LAW IS VALID FOR $5 \times 10^5 < Re_x < 10^7$.

- FROM NOW ON, THE BAR WILL BE REMOVED FROM $\overline{u}$, KNOWING THAT ALL TURBULENT VELOCITIES ARE TIME AVERAGED.

TURBULENT BOUNDARY LAYER -VELOCITY BOUNDARY LAYER

- ALTHOUGH THE "ONE SEVENTH LAW" DESCRIBES WELL THE VELOCITY DISTRIBUTION, IT DOES NOT YIELD THE SHEAR STRESS ON THE WALL:

$$\tau \sim \left(\frac{du}{dy}\right)_{y=0}$$

$$\frac{du}{dy} = \frac{1}{7}\frac{U}{\delta^{1/7}}\frac{1}{y^{6/7}}$$

$$y \to 0 \quad \frac{du}{dy} \to \infty \quad \tau_w \to \infty$$

THIS IS PHYSICALLY NOT ACCEPTABLE.

- IN REALITY "ONE SEVENTH LAW" IS ONLY VALID IN THE BUFFER AND TURBULENT ZONE.
- IN THE LAMINAR SUBLAYER, IT IS ASSUMED THAT THE VELOCITY VARIES LINEARLY.
- THE SLOP OF THIS VARIATION IS SELECTED SUCH THAT IT YIELDS THE WALL SHEAR STRESS OBTAINED EXPERIMENTALLY BY BLASIUS FOR TURBULENT FLOWS ON SMOOTH PLATES:

$$\tau_w = 0.0228\rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4}$$

- THE VELOCITY DISTRIBUTION IN THE LAMINAR SUBLAYER JOINS TO THAT IN THE TURBULENT REGION AT A DISTANCE δ_s.
- δ_s IS CALLED THE THICKNESS OF THE LAMINAR SUBLAYER.

TURBULENT BOUNDARY LAYER -VELOCITY BOUNDARY LAYER

- THE RESULTING VELOCITY PROFILE IS SKETCHED IN THE FOLLOWING FIGURE:

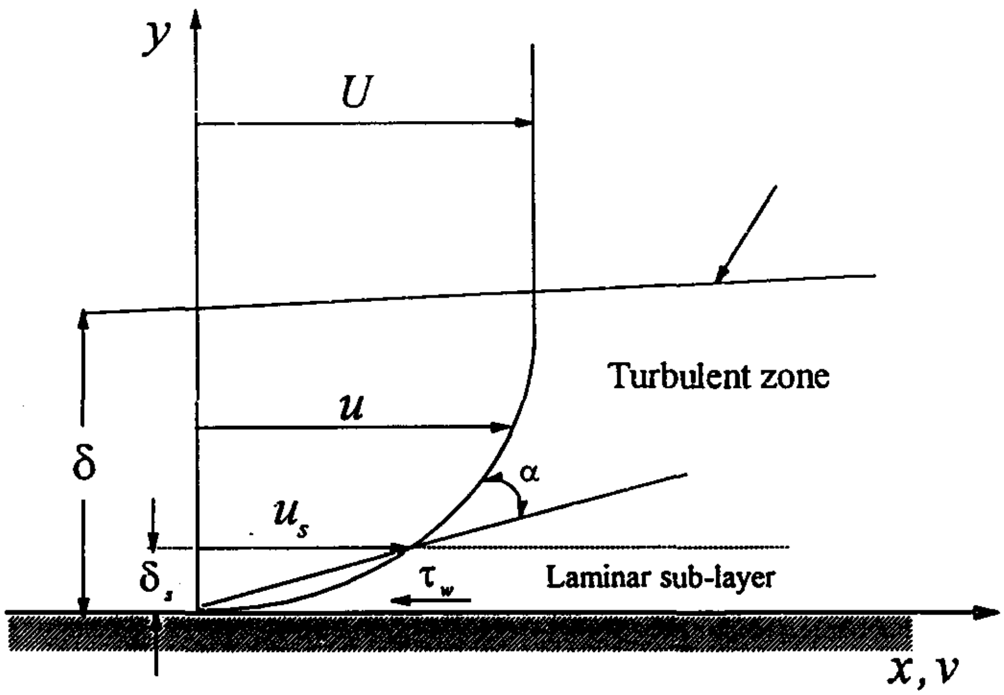

Figure 4.17 Velocity profiles in the turbulent zone and laminar sub-layer.

TURBULENT BOUNDARY LAYER -VELOCITY BOUNDARY LAYER

- TO DETERMINE THE THICKNESS OF THE VELOCITY BOUNDARY LAYER WE WILL USE INTEGRAL MOMENTUM CONSERVATION EQUATION:

$$\rho \frac{d}{dx}\int_0^{\delta(x)} (U(x)-u)u\,dy + \rho \frac{dU(x)}{dx}\int_0^{\delta(x)} (U(x)-u)\,dy = \tau_w$$

$$U(x) = const. = U$$

$$p(x) = const. = p$$

PHYSICAL PROPERTIES ARE CONSTANT.

$$\rho \frac{d}{dx}\int_0^{\delta(x)} (U - u)u\,dy = \tau_w$$

$$\frac{\overline{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

$$\tau_w = 0.0228\rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4}$$

$$\nu = \frac{\mu}{\rho}$$

$$\rho U^2 \frac{d}{dx}\int_0^{\delta} \left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right]dy = 0.0228\rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4}$$

TURBULENT BOUNDARY LAYER -VELOCITY BOUNDARY LAYER

$$\rho U^2 \frac{d}{dx}\int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right]dy = 0.0228\rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4}$$

INTEGRATION

$$\frac{7}{72}\frac{d\delta}{dx} = 0.0228\left(\frac{\nu}{U\delta}\right)^{1/4}$$

or

$$\delta^{1/4} d\delta = 0.235\left(\frac{\nu}{U}\right)^{1/4} dx$$

INTEGRATION

$$\delta = 0.376\left(\frac{\nu}{U}\right)^{1/5} x^{4/5} + const.$$

ASSUMING:
$x = 0 \quad \delta = 0$
(APPROXIMATION)
$const. = 0$

$$\frac{\delta}{x} = \frac{0.376}{\left(\frac{\rho U x}{\mu}\right)^{1/5}} = 0.376\, Re_x^{-1/5}$$

TURBULENT BOUNDARY LAYER -VELOCITY BOUNDARY LAYER

TURBULENT BOUNDARY LAYER - FRICTION COEFFICIENT

IN THE LAMINAR SUBLAYER τ IS GIVEN BY: →

$$\tau = \mu \frac{du}{dy} = \mu \frac{u}{y}$$

$$\tau_w = 0.0228 \rho U^2 \left(\frac{\nu}{U\delta} \right)^{1/4}$$

$$u = 0.0228 \rho \frac{U^2}{\mu} \left(\frac{\mu}{\rho U \delta} \right)^{1/4} y$$

$$y = \delta_s \quad u = u_s$$

$$\frac{\delta_s}{\delta} = \frac{1}{0.0228} \left(\frac{\mu}{\rho U \delta} \right)^{3/4} \frac{u_s}{U}$$

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta} \right)^{1/7}$$

$$y = \delta_s \quad u = u_s$$

$$\frac{\delta_s}{\delta} = \left(\frac{u_s}{U} \right)^7$$

$$\frac{u_s}{U} = 1.878 \left(\frac{\rho U \delta}{\mu} \right)^{-1/8}$$

TURBULENT VELOCITY BOUNDARY LAYER-FRICTION COEFFICIENT

$$\frac{u_s}{U} = 1.878\left(\frac{\rho U \delta}{\mu}\right)^{-1/8}$$

$$\frac{\delta}{x} = \frac{0.376}{\left(\frac{\rho U x}{\mu}\right)^{1/5}} = 0.376\, Re_x^{-1/5}$$

$$\frac{u_s}{U} = 2.12\left(\frac{\mu}{\rho U x}\right)^{0.1} = \frac{2.12}{Re_x^{0.1}}$$

$$\frac{\delta_s}{\delta} = \left(\frac{u_s}{U}\right)^{7}$$

$$\frac{\delta_s}{\delta} = \frac{194}{Re_x^{0.7}}$$

TURBULENT VELOCITY BOUNDARY LAYEP-FRICTION COEFFICIENT

WALL SHEAR STRESS:

$$\tau_w = \mu \frac{u_s}{\delta_s}$$

$$\frac{\delta}{x} = \frac{0.376}{\left(\frac{\rho U x}{\mu}\right)^{1/5}} = 0.376\, Re_x^{-1/5}$$

$$\frac{u_s}{U} = 2.12\left(\frac{\mu}{\rho U x}\right)^{0.1} = \frac{2.12}{Re_x^{0.1}}$$

$$\frac{\delta_s}{\delta} = \frac{194}{Re_x^{0.7}}$$

$$\tau_w = \rho U^2 \frac{0.0296}{Re_x^{0.2}}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

LOCAL WALL FRICTION COEFFICIENT

$$C_f = \frac{0.0592}{Re_x^{0.2}}$$

TURBULENT BOUNDARY LAYER - HEAT TRANSFER COEFFICIENT

- REYNOLDS' ANALOGY
  - LAMINAR BOUNDARY LAYER

SHEAR STRESS AND HEAT FLUX IN A PLANE AT y.

$$\tau = \mu \frac{du}{dy}$$

$$q'' = -k_f \frac{du}{dy}$$

RATIO OF q'' AND τ.

$$\frac{q''}{\tau} = -\frac{k_f}{\mu}\frac{dt}{du}$$

$$\times \frac{c_p}{c_p}$$

$$\frac{q''}{\tau} = -\frac{k_f}{\mu c_p} c_p \frac{dt}{du}$$

$$Pr = \frac{\mu c_p}{k_f}$$

$$\frac{q''}{\tau} = -\frac{1}{Pr} c_p \frac{dt}{du}$$

$$If \quad Pr = 1$$

$$\frac{q''}{\tau} = -c_p \frac{dt}{du}$$

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

$$\frac{q''}{\tau} = -c_p \frac{dt}{du}$$

$$\frac{q''}{\tau} \cong \frac{q''_w}{\tau_w}$$

INTEGRATION

$$\frac{q''_w}{\tau_w} \int_0^U du = -c_p \int_{t_w}^{t_f} dt$$

$$\frac{q''_w}{t_w - t_f} = \frac{\tau_w c_p}{U}$$

$$\frac{q''_w}{t_w - t_f} = h_c$$

$$h_c = \frac{\tau_w c_p}{U}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$h_c = \frac{1}{2}\rho U c_p C_f$$

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

$$h_c = \frac{1}{2}\rho U c_p C_f$$

$$\times x$$

$$Pr = \frac{c_p \mu}{k_f} = 1 \rightarrow c_p = \frac{k_f}{\mu}$$

$$Re_x = \frac{\rho U x}{\mu}$$

$$Nu_x = \frac{h_c x}{k_f}$$

DIMENSIONLESS STATEMENT OF REYNOLDS' ANALOGY FOR LAMINAR FLOW.

$$Nu_x = \frac{1}{2} C_f \, Re_x$$

$$C_f = \frac{0.646}{Re_x^{1/2}}$$

for laminar flow

$$Nu_x = 0.332 \, Re_x^{1/2}$$

COMPARE WITH

$$Nu_x = 0.332 \, Re_x^{1/2} \, Pr^{1/3}$$

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

- IT SEEMS THAT THE EFFECT OF THE PRANDLT NUMBER DIFFERING FROM UNITY CAN BE EXPRESSED BY A FACTOR $Pr^{1/3}$.
- THIS FACT IS SOMETIMES APPLIED TO CASES WHERE EXACT SOLUTION TO THE THERMAL BOUNDARY CANNOT BE OBTAINED; EXPERIMENTAL SKIN FRICTION MEASUREMENTS ARE USED TO PREDICT HEAT TRANSFER COEFFICIENTS.

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

- REYNOLDS' ANALOGY
  - TURBULENT BOUNDARY LAYER.

$$\tau = \tau_l + \tau_t = \rho(\nu + \varepsilon_m)\frac{\partial \bar{u}}{\partial y}$$

$$q'' = -c_p \rho (\alpha + \varepsilon_h)\frac{\partial \bar{t}}{\partial y}$$

ASSUMPTION:

ENTIRE FLOW IN THE BOUNDARY LAYER IS TURBULENT, i.e., LAMINAR SUBLAYER AND BUFFER ZONE ARE IGNORED.

ν KINEMATIC VISCOSITY; RELATED TO DIFFUSIVITY OF MOMENTUM.

ε_m EDDY DIFFUSIVITY OF MOMENTUM. (APPARENT KINEMATIC VISCOSITY).

α MOLECULAR DIFFUSIVITY OF HEAT.

ε_h EDDY DIFFUSIVITY OF HEAT.

$$\nu << \varepsilon_m \; ; \; \alpha << \varepsilon_h$$

$$\varepsilon_m = \varepsilon_h = \varepsilon$$

$$\tau_t = \rho \varepsilon_m \frac{\partial u}{\partial y}$$

$$q_t'' = -c_p \rho \varepsilon_h \frac{\partial t}{\partial y}$$

RATIO

$$\frac{q_t''}{\tau_t} = -c_p \frac{dt}{du}$$

REYNOLDS' ANALOGY FOR TURBULENT FLOW.

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

- PRANDLT'S MODIFICATION TO REYNOLDS' ANALOGY

  - PRANDLT ASSUMES THAT THE TURBULENT BOUNDARY LAYER CONSISTS OF TWO LAYERS:

    1. A VISCOUS LAYER WHERE MOLECULAR DIFFUSIVITY IS DOMINANT:

$$\nu >> \varepsilon_m \quad \textit{and} \quad \alpha >> \varepsilon_h$$

    2. A TURBULENT ZONE WHERE TURBULENT DIFFUSIVITY IS DOMINANT:

$$\varepsilon_m >> \nu \quad \textit{and} \quad \varepsilon_h >> \alpha$$

  - FURTHERMORE, PRANDLT ASSUMES THAT:

$$\varepsilon_m = \varepsilon_h = \varepsilon$$

  - IN THIS APPROACH Pr - NUMBER IS NOT NECESSARILY EQUAL TO 1.

  - THE VARIATION OF VELOCITY AND TEMPERATURE IN THE TWO-REGION BOUNDARY LAYER IS SKETCHED IN THE FOLLOWING FIGURE:

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

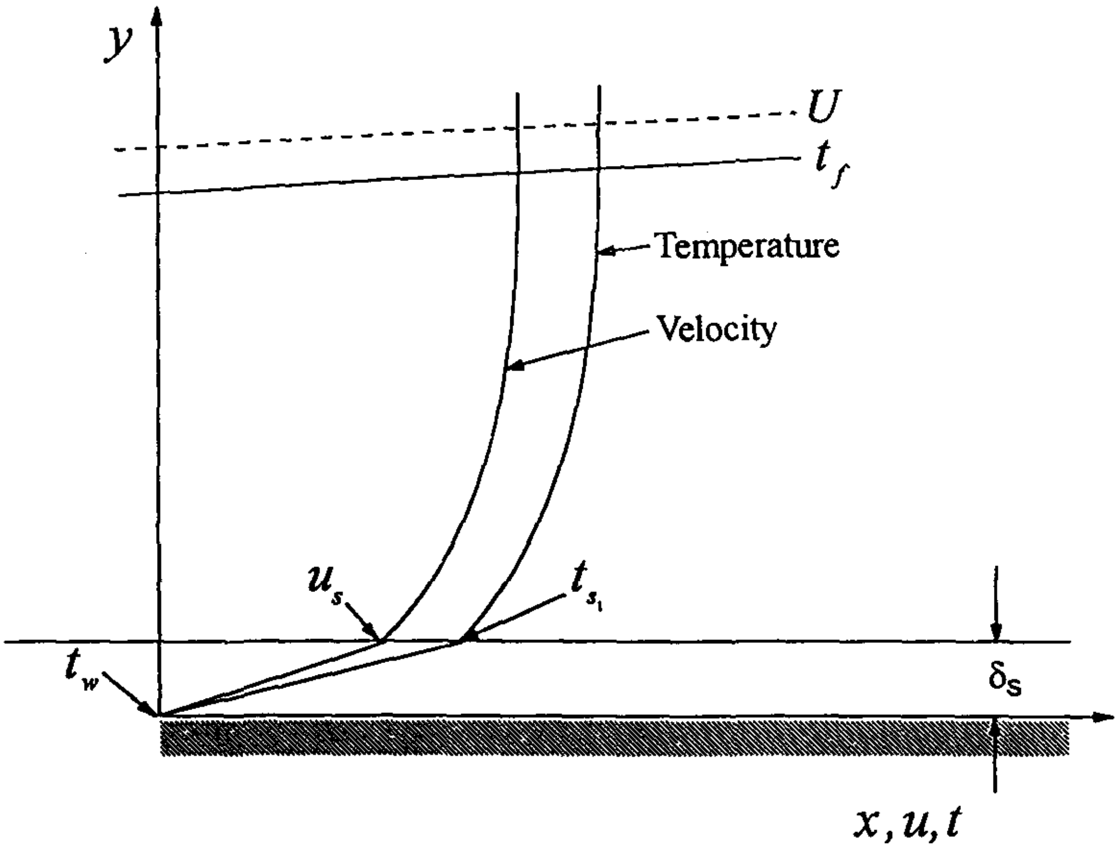

Figure 4.18 Turbulent boundary layer consisting of two zones - Prandlt approach.

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

- LAMINAR SUBLAYER - PRANDLT MODIFICATION

$$\frac{q''}{\tau} = -\frac{k_f}{\mu}\frac{dt}{du}$$

↓

$$\frac{q''}{\tau}du = -\frac{k_f}{\mu}dt$$

→

$$\frac{q''}{\tau} \cong \frac{q''_w}{\tau_w}$$

→

INTEGRATION

$u = 0$ to $u = u_s$

$t = t_w$ to $t = t_{s1}$

↓

$$\frac{q''_w}{\tau_w}\int_0^{u_s} du = -\frac{k_f}{\mu}\int_{t_w}^{t_{s1}} dt$$

↓

$$q''_w = \tau_w \frac{k_f}{\mu}\frac{1}{u_s}(t_w - t_{s1})$$

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

► TURBULENT REGION - PRANDLT MODIFICATION

$$\frac{q''_t}{\tau_t} = -c_p \frac{dt}{du}$$

REYNOLDS' ANALOGY FOR TURBULENT FLOW.

$$\frac{q''_t}{\tau_t} du = -c_p dt$$

$$\frac{q''}{\tau} \cong \frac{q''_w}{\tau_w}$$

INTEGRATION

$u = u_s$ to $u = U$

$t = t_{s1}$ to $t = t_f$

$$\frac{q''_w}{\tau_w} \int_{u_s}^{U} du = -c_p \int_{t_{s1}}^{t_f} dt$$

$$q''_w = \frac{\tau_w c_p}{U - u_s}\left(t_{s1} - t_f\right)$$

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

PRANDLT MODIFICATION

$$q''_w = \tau_w \frac{k_f}{\mu}\frac{1}{u_s}(t_w - t_{s1})$$ LAMINAR SUBLAYER

$$q''_w = \frac{\tau_w c_p}{U - u_s}\left(t_{s1} - t_f\right)$$ TURBULENT REGION

ELIMINATE t_{s1}

$$t_w - t_f = \frac{q''_w}{\tau_w}\left(\frac{\mu u_s}{k_f} + \frac{U - u_s}{c_p}\right)$$

$$q''_w = h_c\left(t_w - t_f\right)$$

$$h = \frac{1}{\dfrac{U}{\tau_w c_p}\left[\dfrac{c_p \mu}{k_f}\dfrac{u_s}{U} + \left(1 - \dfrac{u_s}{U}\right)\right]}$$

$$Pr = \frac{c_p \mu}{k_f}$$

$$h = \frac{\dfrac{\tau_w c_p}{U}}{1 + \dfrac{u_s}{U}(Pr - 1)}$$

THIS IS THE STATEMENT OF PRANDLT'S MODIFY-CATION TO REYNOLDS ANALOGY

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

PRANDLT MODIFICATION

$$h = \frac{\dfrac{\tau_w c_p}{U}}{1 + \dfrac{u_s}{U}(Pr - 1)}$$

$\times \dfrac{x}{k_f}$

REARRANGE

$$Nu_x = \frac{\dfrac{1}{2}\underbrace{\left[\dfrac{\tau_w}{\frac{1}{2}\rho U^2}\right]}_{C_f}\underbrace{\left[\dfrac{\mu c_p}{k_f}\right]}_{Pr}\underbrace{\left[\dfrac{\rho U x}{\mu}\right]}_{Re_x}}{1 + \dfrac{u_s}{U}(Pr - 1)}$$

$$Nu_x = \frac{\dfrac{1}{2} C_f \, Pr \, Re_x}{1 + \dfrac{u_s}{U}(Pr - 1)}$$

$$\frac{u_s}{U} = \frac{2.12}{Re_x^{0.1}} \quad and \quad C_f = \frac{0.0592}{Re_x^{0.2}}$$

$$Nu_x = \frac{0.0292 \, Re_x^{0.8} \, Pr}{1 + 2.12 \, Re_x^{-0.1}(Pr - 1)}$$

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

$$Nu_x = \frac{0.0292\, Re_x^{0.8}\, Pr}{1 + 2.12\, Re_x^{-0.1}(Pr - 1)}$$

- THIS IS THE CONVECTION HEAT TRANSFER CORRELATION FOR A TURBULENT FLOW OVER A FLAT PLATE.

- APPLICATION CONDITIONS:

  - FLUID PROPERTIES MUST BE EVALUATED AT THE MEAN BOUNDARY LAYER TEMPERATURE.

$$t_m = \frac{t_w + t_f}{2}$$

  - $Pr \cong 1$

- THE ABOVE CORRELATION IS DIFFICULT TO INTEGRATE.

- THE FOLLOWING CORRELATION GIVE GOOD RESULTS:

$$Nu_x = 0.0292\, Re_x^{0.8}\, Pr^{1/3}$$

DETERMINE FLUID PROPERTIES AT THE MEAN BOUNDARY LAYER TEMPERATURE.

$$\bar{h}_c = \frac{\int_0^L h_{cx}\, dx}{\int_0^L dx}$$

$$\bar{h}_c = \frac{1}{L}\, 0.0292\, Pr^{1/3} \left(\frac{\rho U}{\mu}\right)^{0.8} k_f \int_0^L \frac{1}{x^{0.2}}\, dx$$

or

$$Nu_L = \frac{\bar{h}_c L}{k_f} = 0.036\, Re_L^{0.8}\, Pr^{1/3}$$

TURBULENT BOUNDARY LAYER - LOCAL HEAT TRANSFER COEFFICIENT

$$Nu_L = \frac{\bar{h}_c L}{k_f} = 0.036\, Re_L^{0.8}\, Pr^{1/3}$$

- THE ABOVE CORRELATION ASSUMES THAT THE BOUNDARY LAYER IS TURBULENT STARTING FROM THE LEADING EDGE OF THE PLATE.
- HOWEVER, WE KNOW THAT A PORTION OF THE PLATE IS OCCUPIED BY A LAMINAR BOUNDARY LAYER; THE REST BY TURBULENT BOUNDARY LAYER.
- THE AVERAGE HEAT TRANSFER COEFFICIENT INCLUDING BOTH REGIONS IS THAN GIVEN BY:

$$\bar{h}_c = \frac{\int_0^{x_c} h_{cl}\,dx + \int_{x_c}^{L} h_{ct}\,dx}{L}$$

$$Nu_x = 0.332\, Re_x^{1/2}\, Pr^{1/3}$$
$$Nu_x = 0.0292\, Re_x^{0.8}\, Pr^{1/3}$$

$$\bar{h}_c = \frac{1}{L}\left[0.332\,Pr^{1/3}\,k_f\left(\frac{\rho U}{\mu}\right)^{1/2}\int_0^{x_c}\frac{1}{x^{1/2}}\,dx + 0.0292\,Pr^{1/3}\,k_f\left(\frac{\rho U}{\mu}\right)^{0.8}\int_{x_c}^{L}\frac{1}{x^{0.2}}\,dx\right]$$

$$Nu_L = 0.036\,Pr^{1/3}\left[Re_L^{0.8} - Re_{cr}^{0.8} + 18.44\,Re_{cr}^{1/2}\right]$$

$$Re_{cr} = 5.\times 10^5$$

$$Nu_L = 0.036\,Pr^{1/3}\left[Re_L^{0.8} - 23{,}100\right]$$

FORCED CONVECTION INSIDE DUCTS

- HEATING AND COOLING OF FLUIDS FLOWING INSIDE A DUCT CONSTITUTE ONE OF THE MOST FREQUENTLY ENCOUNTERED ENGINEERING PROBLEMS.

- FLOW INSIDE A DUCT CAN BE:
  - LAMINAR, OR
  - TURBULENT.

- TURBULENT FLOWS ARE THE MOST WIDELY ENCOUNTERED TYPE IN THE INDUSTRIAL APPLICATIONS.

- WHEN A FLUID WITH UNIFORM VELOCITY ENTERS A STRAIGHT PIPE A VELOCITY BOUNDARY LAYER STARTS DEVELOPING.

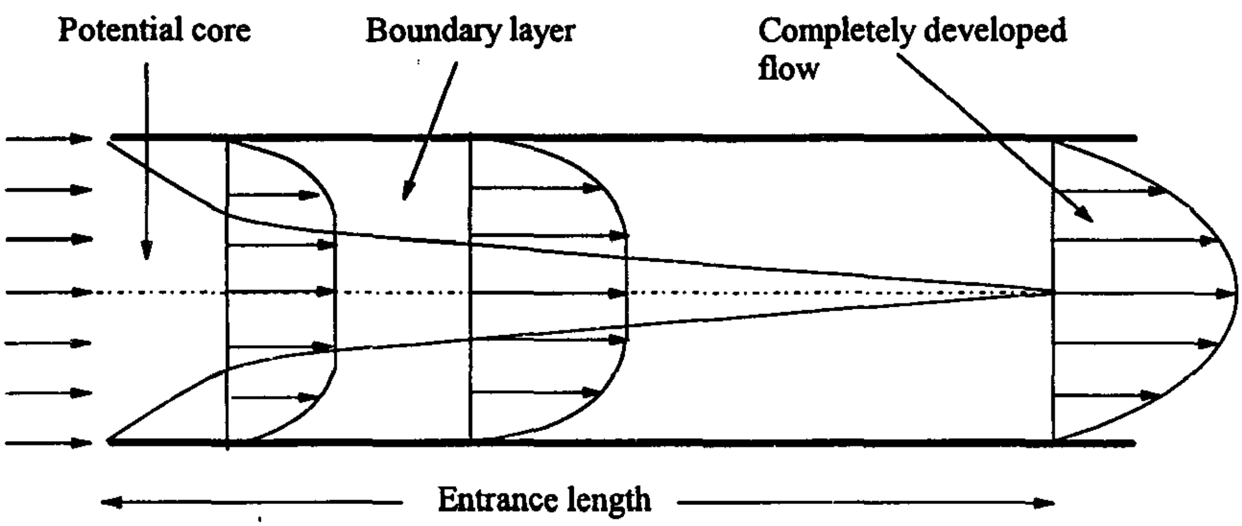

Figure 4.19 Flow in the entrance region of a pipe.

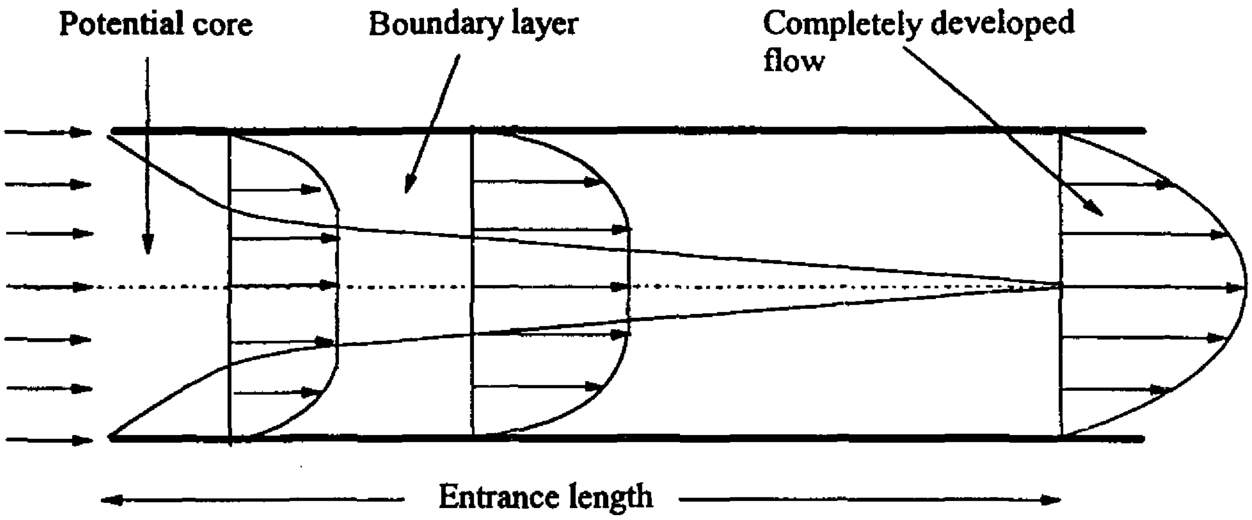

Figure 4.19 Flow in the entrance region of a pipe.

- AS WE PROCEED ALONG THE TUBE IN THE ENTRANCE REGION, THE PORTION OF THE TUBE OCCUPIED
  - BY THE BOUNDARY LAYER INCREASES, AND
  - THAT OCCUPIED BY THE POTENTIAL FLOW DECREASES.

- CONSEQUENTLY, TO SATISFY THE MASS CONSERVATION PRINCIPLE, i.e., CONSTANT AVERAGE VELOCITY,
  - THE VELOCITY OF THE POTENTIAL CORE SHOULD INCREASE.

- THE TRANSITION FROM LAMINAR FLOW TO TURBULENT FLOW IS LIKELY TO OCCUR IN THE ENTRANCE LENGTH.

- IF THE BOUNDARY IS LAMINAR UNTIL IT FILLS THE TUBE, THE FLOW IN THE FULL DEVELOPED REGION WILL BE LAMINAR WITH A PARABOLIC VELOCITY PROFILE.

LAMINAR FLOW IN DUCTS - VELOCITY DISTRIBUTION IN FULLY DEVELOPED REGION

- THE VELOCITY DISTRIBUTION CAN EASILY DETERMINED FOR A STEADY STATE LAMINAR FLOW IN THE FULLY DEVELOPED REGION.
- IN THIS REGION, VELOCITY PROFILE DOES NOT CHANGE ALONG THE TUBE.
- IT DEPENDS ONLY ON THE RADIUS, i.e., $u = u(r)$.

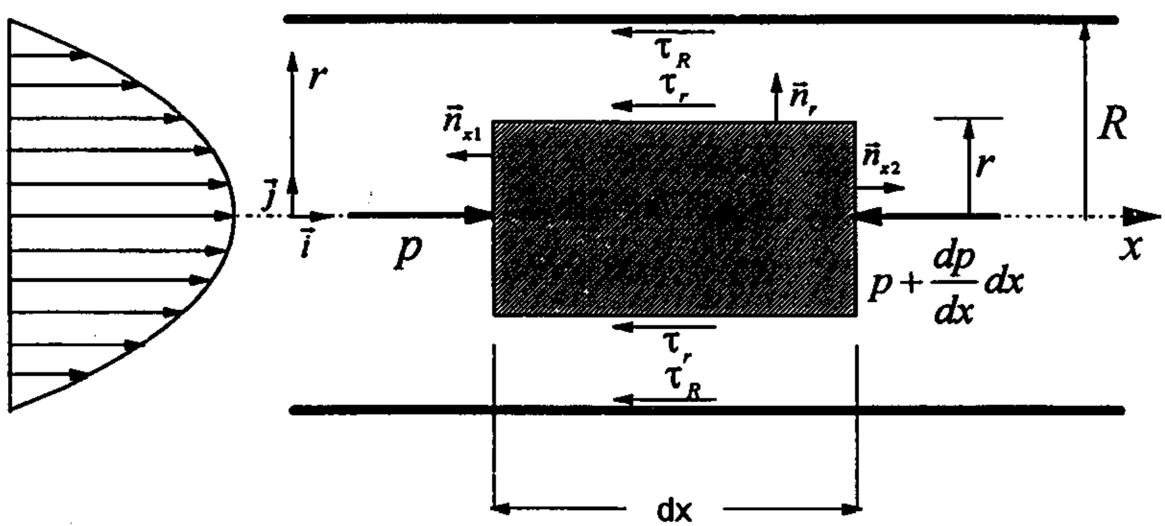

Figure 4.21 Control volume in a laminar, fully developed flow in a circular tube

LAMINAR FLOWS IN DUCTS - VELOCITY DISTRIBUTION

$$\frac{d}{d\tau}\int_{V(\tau)} \rho\vec{v}\,dV = -\int_{A(\tau)} \vec{n}.\rho\vec{v}(\vec{v}-\vec{\omega})\,dA - \int_{A(\tau)} \vec{n}.p\bar{\bar{I}}\,dA + \int_{A(\tau)} \vec{n}.\bar{\bar{\sigma}}\,dA + \int_{V(\tau)} \rho\vec{g}\,dV$$

- STEADY STATE
- $\vec{\omega} = 0$
- GRAVITY NEGLECTED

$$-\int_A \vec{n}.\rho\vec{v}\vec{v}\,dA - \int_A \vec{n}.p\bar{\bar{I}}\,dA + \int_A \vec{n}.\bar{\bar{\sigma}}\,dA = 0$$

$$\int_A \vec{n}.\rho\vec{v}\vec{v}\,dA = 0$$

$$-\int_A \vec{n}.p\bar{\bar{I}}\,dA + \int_A \vec{n}.\bar{\bar{\sigma}}\,dA = 0$$

$$\times\vec{i}$$

$$-\vec{i}\cdot\int_A \vec{n}.p\bar{\bar{I}}\,dA + \vec{i}\cdot\int_A \vec{n}.\bar{\bar{\sigma}}\,dA = 0$$

LAMINAR FLOWS IN DUCTS - VELOCITY DISTRIBUTION

$$-\vec{i}\cdot\int_A \vec{n}.\,p\overline{\overline{I}}d\,A+\vec{i}\cdot\int_A \vec{n}.\overline{\overline{\sigma}}d\,A=0$$

$$\vec{i}\cdot\int_A \vec{n}.\,p\overline{\overline{I}}d\,A=-p_x\pi r^2+\left(p_x+\frac{dp_x}{dx}dx\right)\pi r^2$$

$$=\pi r^2\frac{dp}{dx}dx$$

$$\vec{i}\cdot\int_A \vec{n}.\overline{\overline{\sigma}}d\,A=-2\pi r dx\tau_r$$

$$\frac{r}{2}\frac{dp}{dx}=-\tau_r$$

$$\tau_r=\mu\frac{du}{dy}$$

$$y=R-r$$
$$dy=-dr$$

$$\tau_r=-\mu\frac{du}{dr}$$

$$du=\frac{1}{2\mu}\left(\frac{dp}{dx}\right)r\,dr$$

(dp/dx) INDEPENDENT OF r.

INTEGRATION

$$u=\frac{1}{4\mu}\left(\frac{dp}{dx}\right)r^2+C$$

LAMINAR FLOWS IN DUCTS - VELOCITY DISTRIBUTION

$$u = \frac{1}{4\mu}\left(\frac{dp}{dx}\right) r^2 + C$$

$r = R \quad u = 0$

$$C = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right) R^2$$

$$u = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right) R^2 \left(1 - \frac{r^2}{R^2}\right) = U_{max}\left(1 - \frac{r^2}{R^2}\right)$$

$$U_m = \frac{\int_0^R 2\pi r u dr}{\pi R^2}$$

$$= -\frac{R^2}{8\mu}\left(\frac{dp}{dx}\right) = \frac{1}{2} U_{max}$$

$$u = 2U_m\left(1 - \frac{r^2}{R^2}\right)$$

$$U_m = \frac{Q}{A}$$

Q : VOLUME FLOW RATE.

A : FLOW SECTION

LAMINAR FLOW IN DUCTS - PRESSURE DROP AND FRICTION FACTOR

- CONSIDER NOW A CONTROL VOLUME BOUNDED BY THE TUBE WALL AND TWO PLANES PERPENDICULAR TO THE AXIS AND A DISTANCE *dx* APART.

$$\frac{r}{2}\frac{dp}{dx} = -\tau_r$$

SELECTED CONTROL VOLUME

$$\frac{R}{2}\frac{dp}{dx} = -\tau_R$$

or

$$\frac{dp}{dx} = -\frac{4\tau_R}{D}$$

- DIVIDE BY: $\frac{1}{2}\rho U_m^2$
- DEFINE FRICTION FACTOR AS:

$$f = \frac{4\tau_R}{\frac{1}{2}\rho U_m^2}$$

$$-\frac{dp}{dx} = f\,\frac{1}{2}\rho U_m^2\,\frac{1}{D}$$

LAMINAR FLOWS IN DUCTS - PRESSURE DROP AND FRICTION FACTOR

► FRICTION FACTOR

$$\tau_R = -\mu\left(\frac{du}{dr}\right)_{r=R}$$

$$u = 2U_m\left(1-\frac{r^2}{R^2}\right)$$

$$\frac{du}{dr} = -\frac{4U_m}{R} = -\frac{8U_m}{D}$$

$$\tau_R = \frac{8\mu U_m}{D}$$

$$f = \frac{4\tau_R}{\frac{1}{2}\rho U_m^2}$$

$$f = \frac{64\mu}{\rho U_m D}$$

$$Re_D = \frac{\rho U_m D}{\mu}$$

$$f = \frac{64}{Re_D}$$

FRICTION FACTOR
FOR LAMINAR
FLOWS IN TUBES

LAMINAR FLOWS IN DUCTS - PRESSURE DROP AND FRICTION FACTOR

- TOTAL PRESSURE DROP IN A TUBE OF LENGTH *L*.

$$-\frac{dp}{dx} = f\,\frac{1}{2}\rho U_m^2\,\frac{1}{D}$$

INTEGRATION BETWEEN THE ENTRANCE AND EXIT OF THE TUBE

$$\Delta P = -\int_{p_1}^{p_2} dp = \int_0^L f\,\frac{1}{2}\rho U_m^2\,\frac{1}{D}\,dx = f\,\frac{1}{2}\rho U_m^2\,\frac{L}{D}$$

$$f = \frac{64}{Re_D}$$

BULK TEMPERATURE

- FOR FLOW OVER A FLAT PLATE, THE CONVECTION HEAT TRANSFER COEFFICIENT WAS DEFINED AS:

$$h_c = \frac{q''}{t_w - t_f}$$

t_f IS THE POTENTIAL STREAM TEMPERATURE.

- IN A TUBE FLOW, THERE IS NO DISCERNIBLE FREE STREAM CONDITION.

- THE CENTERLINE TEMPERATURE OF A TUBE FLOW IS NOT EASILY DETERMINABLE.

- CONSEQUENTLY, FOR A FULLY DEVELOPED PIPE FLOW IT IS CUSTOMARY TO DEFINE A "**BULK TEMPERATURE**" AS:

$$t_b = \frac{\int_0^R \rho c_p t u 2\pi r dr}{\int_0^R \rho c_p u 2\pi r dr}$$

  - THE NUMERATOR REPRESENTS THE TOTAL ENERGY FLOW THROUGH THE PIPE.

  - THE DENOMINATOR REPRESENTS THE PRODUCT OF THE MASS FLOW AND THE SPECIFIC HEAT INTEGRATED OVER THE FLOW AREA.

LAMINAR FLOWS IN DUCTS - BULK TEMPERATURE

- WITH THE DEFINITION OF THE "BULK TEMPERATURE" THE LOCAL HEAT TRANSFER COEFFICIENT IN A PIPE FLOW IS GIVEN BY:

$$h_c = \frac{q''}{t_w - t_b}$$

- IN PRACTICE, IN A HEATED TUBE, AN ENERGY BALANCE MAY BE USED TO DETERMINE THE BULK TEMPERATURE AND ITS VARIATION ALONG THE TUBE.

- TWO CASES WILL BE CONSIDERED:

  1. CONSTANT SURFACE HEAT FLUX.
  2. CONSTANT SURFACE TEMPERATURE.

LAMINAR FLOWS IN DUCTS - DETERMINATION OF THE BULK TEMPERATURE

- DETERMINATION OF THE BULK TEMPERATURE BY ENERGY BALANCE.

  ► CONSTANT SURFACE HEAT FLUX

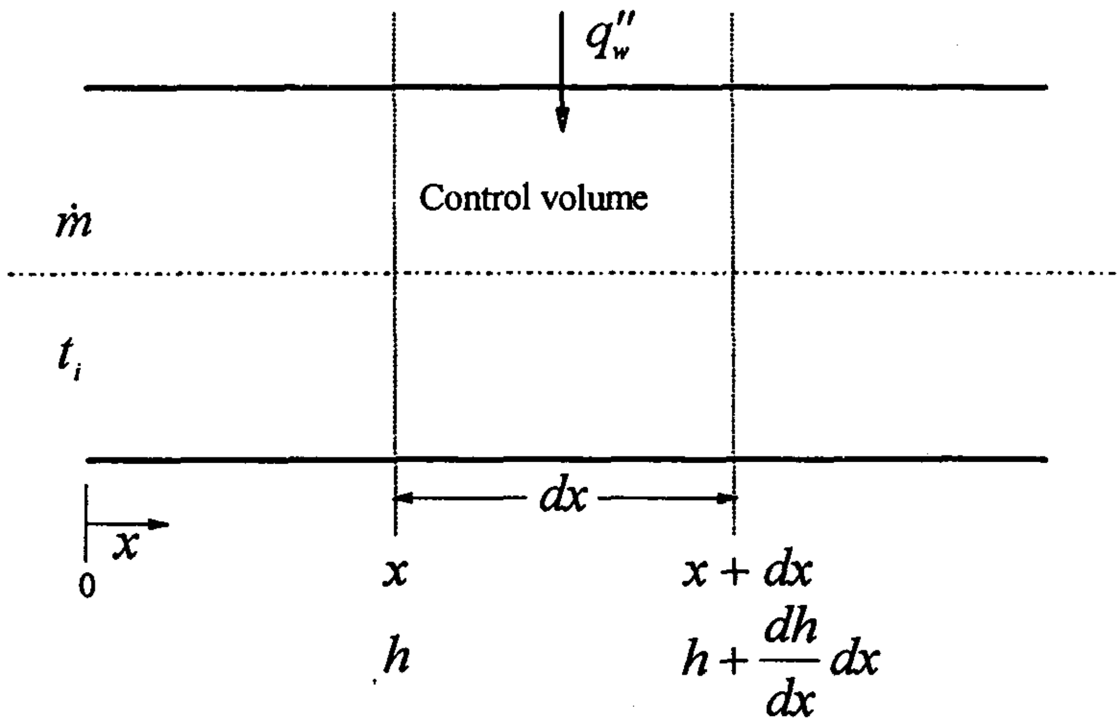

Figure 4.22 Control volume for internal flow in a tube.

$\dot{m}$: MASS FLOW RATE

t_i : INLET TEMPERATURE

h_i : INLET ENTHALPY

- KINETIC END POTENTIAL ENERGIES, VISCOUS DISSIPATION AND AXIAL HEAT CONDUCTION ARE NEGLIGIBLE.

LAMINAR FLOWS IN DUCTS - DETERMINATION OF THE BULK TEMPERATURE

► (CONSTANT SURFACE HEAT FLUX)

ENERGY CONSERVATION EQUATION

$$\int_A \vec{n}.\rho h\vec{v}\,dA + \int_A \vec{n}.\vec{q}''\,dA = 0$$

SELECTED CONTROL VOLUME IN THE ABOVE FIGURE

$$q_w''\pi D dx = \dot{m}\left(h + \frac{dh}{dx}dx - h\right)$$

$$dh = \frac{\pi D}{\dot{m}} q_w'' dx$$

INTEGRATION:

$h = h_i$ and $h = h$

$x = 0$ and $x = x$

$$h(x) - h_i = \frac{\pi D}{\dot{m}} q_w'' x$$

$$h(x) - h_i = \bar{c}_p\left(t_b(x) - t_i\right)$$

$$t_b(x) = t_i + \frac{\pi D}{\dot{m}\bar{c}_p} q_w'' x$$

LAMINAR FLOWS IN DUCTS - DETERMINATION OF THE BULK TEMPERATURE

► CONSTANT SURFACE TEMPERATURE

$$dh = \frac{\pi D}{\dot{m}} q''_w dx$$

$$q''_w(x) = h_c\left[t_w - t_b(x)\right]$$

$$dh = c_p dt$$

$$\frac{dt}{t_w - t_b(x)} = \frac{\pi D}{\dot{m}} \frac{h_c}{c_p} dx$$

INTEGRATION

$$\ln\left(t_w - t_b(x)\right) = -\frac{\pi D}{\dot{m}\overline{c}_p} \int_0^x h_c dx + C'$$

$$t_w - t_b(x) = \exp C' \cdot \exp\left(-\frac{\pi D}{\dot{m}\overline{c}_p} \int_0^x h_c dx\right)$$

$$t_w - t_b(x) = C \exp\left(-\frac{\pi D}{\dot{m}\overline{c}_p} \int_0^x h_c dx\right)$$

LAMINAR FLOWS IN DUCTS - DETERMINATION OF THE BULK TEMPERATURE

- (CONSTANT SURFACE TEMPERATURE)

$$t_w - t_b(x) = C\exp\left(-\frac{\pi D}{\dot{m}\bar{c}_p}\int_0^x h_c dx\right)$$

BOUNDARY CONDITIONS

$x = 0 \quad t_w - t_b = t_w - t_i$

$C = t_w - t_i$

$$\frac{t_w - t_b(x)}{t_w - t_i} = \exp\left(-\frac{\pi D}{\dot{m}\bar{c}_p}\int_0^x h_c dx\right)$$

DEFINING:

$\bar{h}_c = \frac{1}{x}\int_0^x h_c dx$

$$\frac{t_w - t_b(x)}{t_w - t_i} = \exp\left(-\frac{\pi D x}{\dot{m}\bar{c}_p}\bar{h}_c\right)$$

TEMPERATURE AT THE EXIT OF THE TUBE:

$x = L$

$$\frac{t_w - t_e}{t_w - t_i} = \exp\left(-\frac{\pi D L}{\dot{m}\bar{c}_p}\bar{h}_{cL}\right)$$

LAMINAR FLOWS IN DUCTS - DETERMINATION OF THE BULK TEMPERATURE

- IF h_c CAN BE TAKEN AS CONSTANT ALONG THE TUBE, THE DETERMINATION OF $t_b(x)$ IS STRAIGHT FORWARD.
- IF NOT, ITERATIONS ARE REQUIRED TO DETERMINE THE VALUE OF THE BULK TEMPERATURE.

- BULK TEMPERATURE CONCEPT INTRODUCED HERE IS APPLICABLE TO BOTH LAMINAR AND TURBULENT FLOWS.

LAMINAR FLOW IN DUCTS - HEAT TRANSFER COEFFICIENT

- CONTRARILY TO VELOCITY DISTRIBUTION, ANALYTICAL INVESTIGATION OF THE TEMPERATURE DISTRIBUTION AND ,CONSEQUENTLY, THE CONVECTION HEAT TRANSFER COEFFICIENT IS COMPLEX.

- IN A CIRCULAR TUBE WITH UNIFORM WALL HEAT FLUX AND FULLY DEVELOPED LAMINAR FLOW, IT IS ANALYTICALLY FOUND THAT:

$$Nu_D = \frac{h_c D}{k_f} = 4.364$$

  i.e., Nu_D IS INDEPENDENT OF Re_D , Pr AND AXIAL LOCATION.

  - IN THIS ANALYSIS, IT IS ASSUMED THAT THE VELOCITY DISTRIBUTION IS GIVEN BY THAT CORRESPONDING TO ISOTHERMAL FLUID FLOWS.

- FOR CONSTANT WALL TEMPERATURE CONDITION, IT IS FOUND THAT:

$$Nu_D = \frac{h_c D}{k_f} = 3.66$$

  - AGAIN ISOTHERMAL FLUID FLOW VELOCITY DISTRIBUTION IS USED.

- THE USE OF A VELOCITY DISTRIBUTION CORRESPONDING TO ISOTHERMAL FLUID FLOW CONDITION IS ONLY VALID FOR SMALL TEMPERATURE DIFFERENCE BETWEEN THE FLUID AND WALL TEMPERATURE.

- FOR LARGE TEMPERATURE DIFFERENCES, THE FLUID VELOCITY IS INFLUENCED BY THESE DIFFERENCES AS SKETCHED IN THE FOLLOWING FIGURE:

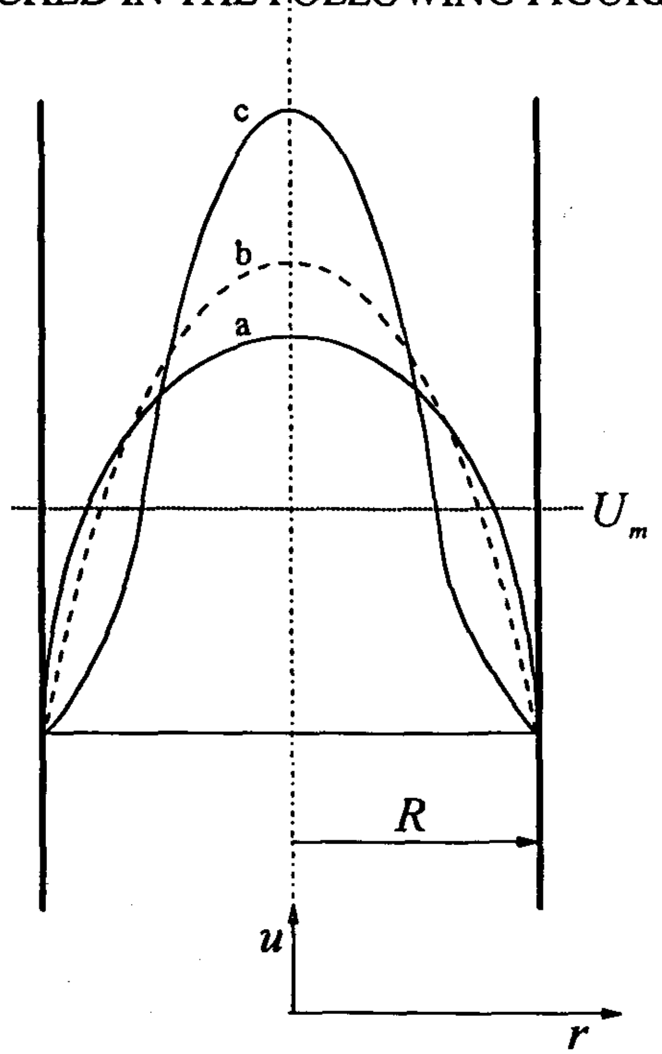

Figure 4.23 Influence of large temperature differences on velocity distribution in a tube

LAMINAR FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

- CURVE (b) IS THE VELOCITY DISTRIBUTION FOR AN ISOTHERMAL OR SMALL TEMPERATURE DIFFERENCE FLOW.

- CURVE (a) IS THE VELOCITY DISTRIBUTION WHEN THE WALL HEATS A LIQUID OR COOLS A GAS.

- CURVE (c) IS THE VELOCITY DISTRIBUTION WHEN THE WALL COOLS A LIQUID OR HEATS A GAS.

- THE ABOVE PRESENTED HEAT TRANSFER CORRELATIONS ARE ENTICING BY THEIR SIMPLICITY.

- HOWEVER, BECAUSE OF THE VELOCITY PROFILE CHANGES DUE TO HEATING OR COOLING THEY ARE NOT ACCURATE.

- THESE CORRELATIONS ARE ONLY APPLICABLE TO FULLY DEVELOPED FLOWS.

- HOWEVER, THE LENGTH OF THE ENTRANCE REGION IN A LAMINAR FLOW IS SUBSTANTIAL; IT MAY EVEN OCCUPY THE ENTIRE LENGTH OF THE TUBE.

- THE FOLLOWING CORRELATION PREDICTS THE CONVECTION HEAT TRANSFER COEFFICIENT IN THE ENTRANCE REGION.

$$\overline{Nu_D} = \frac{\bar{h}_c D}{k_f} = 3.66 + \frac{0.0668(D/L)Re_D Pr}{1 + 0.04\left[(D/L)\,Re_D\,Pr\right]^{2/3}}$$

LAMINAR FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

$$\overline{Nu_D} = \frac{\overline{h}_c D}{k_f} = 3.66 + \frac{0.0668(D/L)Re_D Pr}{1 + 0.04[(D/L)\,Re_D\,Pr]^{2/3}}$$

- $\overline{Nu_D}$ IS THE AVERAGE NUSSELT NUMBER.
- AS THE PIPE LENGTH INCREASES, THIS CORRELATION TENDS TO 3.66.
- FLUID PROPERTIES ARE CALCULATED AT THE BULK TEMPERATURE.
- THIS CORRELATION IS VALID FOR:

$$\left(\frac{D}{L}\right) Re_D\, Pr < 100$$

- A BETTER CORRELATION FOR LAMINAR FLOWS (SIEDER AND TATE) IS:

$$\overline{Nu_D} = 1.86\, Re_D^{1/3}\, Pr^{1/3}\left(\frac{D}{L}\right)^{1/3}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

- FLUID PROPERTIES (EXCEPT μ_w) ARE EVALUATED AT THE BULK TEMPERATURE.
- μ_w IS EVALUATED AT THE WALL TEMPERATURE.
- THE TERM, $(\mu_b/\mu_w)^{0.14}$ TAKES INTO ACCOUNT THE FACT THAT THE BOUNDARY LAYER AT THE WALL IS STRONGLY INFLUENCED BY THE TEMPERATURE DEPENDENCE OF THE VISCOSITY.
- $(\mu_b/\mu_w)^{0.14}$ APPLIES FOR HEATING AND COOLING CASES.
- THE EFFECT OF THE ENTRANCE LENGTH IS INCLUDED IN THE TERM $(D/L)^{1/3}$.

LAMINAR FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

$$\overline{Nu_D} = 1.86\, Re_D^{1/3}\, Pr^{1/3} \left(\frac{D}{L}\right)^{1/3} \left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

- THE RANGE OF APPLICABILITY:

$$0.48 < Pr < 16{,}700$$

$$0.0044 < \frac{\mu_b}{\mu_w} < 9.75$$

$$\left(\frac{Re_D\, Pr}{L/D}\right)^{1/3} \left(\frac{\mu_b}{\mu_w}\right)^{0.14} \geq 2$$

TURBULENT FLOWS IN DUCTS.

- IT IS EXPERIMENTALLY VERIFIED THAT:

  - ONE SEVENTH LAW: $\frac{\overline{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7}$

  - BLASIUS RELATION: $\tau_w = 0.0228 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4}$

  - $\frac{\delta_s}{\delta}$ RATIO: $\frac{\delta_s}{\delta} = \frac{1}{0.0228}\left(\frac{\mu}{\rho U \delta}\right)^{3/4} \frac{u_s}{U}$

  - $\frac{u_s}{u}$ RATIO: $\frac{u_s}{U} = 1.878\left(\frac{\rho U \delta}{\mu}\right)^{-1/8}$

ESTABLISHED FOR A TURBULENT BOUNDARY LAYER ON A FLAT PLATE CAN BE EXTENDED TO FULLY DEVELOPED TURBULENT FLOWS IN SMOOTH TUBES.

TURBULENT FLOW IN DUCTS - VELOCITY DISTRIBUTION IN FULLY DEVELOPED REGION

ONE SEVENTH LAW FOR A TURBULENT FLOW OVER A FLAT PLATE

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

$$y \rightarrow R - r$$

$$\delta \rightarrow R \; or \; D/2$$

$$U \rightarrow U_{max}$$

$$\frac{u}{U_{max}} = \left(\frac{R-r}{R}\right)^{1/7}$$

$$U_m = \frac{\int_0^R 2\pi r u dr}{\pi R^2}$$

$$U_m = \frac{U_{max}}{\pi R^2} \int_0^R \left(\frac{R-r}{R}\right)^{1/7} dr = 0.817 U_{max}$$

TURBULENT FLOW IN A DUCTS - FRICTION FACTOR AND FRICTIONAL PRESSURE GRADIENT

- FRICTION FACTOR

BLASIUS CORRELATION FOR A TURBULENT FLOW ON A FLAT PLATE.

$$\tau_w = 0.0228\rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4}$$

$$\tau_w \rightarrow \tau_R$$
$$\delta \rightarrow D/2$$
$$U \rightarrow U_{max} = \frac{U_m}{0.817}$$

$$\tau_R = 0.039\rho U_m^2\left(\frac{\nu}{U_m D}\right)^{1/4}$$

$$f = \frac{4\tau_R}{\frac{1}{2}\rho U_m^2}$$

$$f = \frac{0.312}{\left(\frac{U_m D}{\nu}\right)^{1/4}} = \frac{0.312}{Re_D^{1/4}}$$

- VALID FOR:
  $$10^4 < Re_D < 5\times10^4$$
- IF 0.312 IS REPLACED BY 0.316 THE CORRELATION IS THEN VALID FOR:
  $$10^4 < Re_D < 10^5$$

$$Re_D = \frac{U_m D}{\nu}$$

TURBULENT FLOWS IN DUCTS - FRICTION FACTOR AND PRESSURE GRADIENT

- OTHER FRICTION CORRELATIONS

  - PRANDLT CORRELATION:

$$\frac{1}{\sqrt{f}} = 2.0\log\left(Re\sqrt{f}\right) - 0.8 \qquad 3{,}000 < Re_D < 3.4\times10^6$$

  - VON KARMAN CORRELATION:

$$\frac{1}{\sqrt{f}} = 2.0\log\left(\frac{D}{\varepsilon}\right) + 1.74 \qquad \frac{D}{\varepsilon}\frac{1}{Re_D\sqrt{f}} > 0.01$$

ε IS THE RUGOSITY OF THE TUBE WALL.

- FRICTIONAL PRESSURE DROP GRADIENT:

$$-\frac{dp}{dx} = f\,\frac{1}{2}\rho U_m^2\,\frac{1}{D}$$

TURBULENT FLOWS IN DUCTS - CONVECTION HEAT TRANSFER COEFFICIENT

- HEAT TRANSFER COEFFICIENT ESTABLISHED FOR A FLAT PLATE:

$$h = \frac{\dfrac{\tau_w c_p}{U}}{1 + \dfrac{u_s}{U}(Pr - 1)}$$

WILL BE APPLIED TO TURBULENT FLOWS IN PIPES WITH SOME MODIFICATIONS.

TURBULENT FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT
(CONVECTION HEAT TRANSFER COEFFICIENT)

$$h = \frac{\dfrac{\tau_w C_p}{U}}{1 + \dfrac{u_s}{U}(Pr - 1)}$$

$$U \rightarrow U_m$$

$$\frac{u_s}{U} = 1.878\left(\frac{\rho U \delta}{\mu}\right)^{-1/8}$$

$$U \rightarrow U_{max}$$

$$U_{max} = \frac{U_m}{0.817}$$

$$\delta = \frac{D}{2}$$

$$\frac{u_s}{U_m} = 2.44\left(\frac{\mu}{\rho U_m D}\right)^{1/8}$$

or

$$\frac{u_s}{U_m} = \frac{2.44}{Re_D^{1/8}}$$

TURBULENT FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

(CONVECTION HEAT TRANSFER COEFFICIENT)

$$h = \frac{\dfrac{\tau_w c_p}{U}}{1 + \dfrac{u_s}{U}(Pr - 1)}$$

$$\tau_w \rightarrow \tau_R$$

$$\frac{4\tau_R}{\frac{1}{2}\rho U_m^2} = f$$

$$f = \frac{0.316}{Re_D^{0.25}}$$

$$\tau_R = \frac{1}{8}(\rho U_m^2)\frac{0.316}{Re_D^{0.25}}$$

$$Nu_D = \frac{h_c D}{k_f}$$

$$Re_D = \frac{\rho U_m D}{\mu}$$

$$Nu_D = \frac{0.0396\, Re_D^{3/4}\, Pr}{1 + 2.44\, Re_D^{-1/8}(Pr - 1)}$$

TURBULENT FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

(CONVECTION HEAT TRANSFER COEFFICIENT)

$$Nu_D = \frac{0.0396\, Re_D^{3/4}\, Pr}{1 + 2.44\, Re_D^{-1/8} (Pr - 1)}$$

- THIS CORRELATION WORKS REASONABLY WELL.
- IT IS BETTER TO REPLACE:

$$2.44 \quad by \quad 1.5\, Pr^{-1/6}$$

i.e.,

$$Nu_D = \frac{0.0396\, Re_D^{3/4}\, Pr}{1 + 1.5\, Pr^{-1/6}\, Re_D^{-1/8} (Pr - 1)}$$

- ► FLUID PROPERTIES ARE DETERMINED AT THE BULK FLUID TEMPERATURE.
- ► *Pr* NUMBER SHOULD BE CLOSE TO 1.

- OTHER CONVECTION HEAT TRANSFER CORRELATIONS FOR TURBULENT FLOWS IN PIPES.

  - IF $(t_w - t_b)$ IS LESS THAN 6 °C FOR LIQUIDS OR 60 °C FOR GASES, USE THE FOLLOWING DITTIUS-BOELTER CORRELATION:

$$Nu_D = 0.023\, Re_D^{0.8}\, Pr^n$$

$n = 0.4$ FOR HEATING,

$n = 0.3$ FOR COOLING.

- FLUID PROPERTIES ARE DETERMINED AT THE BULK TEMPERATURE.

- RANGE OF APPLICABILITY:

$$0.7 < Pr < 160$$

$$Re_D > 10{,}000$$

$$\frac{L}{D} > 60$$

TURBULENT FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

- OTHER CONVECTION HEAT TRANSFER CORRELATIONS FOR TURBULENT FLOWS IN PIPES.

  ► IF $(t_w - t_b)$ IS HIGHER THAN 6 °C FOR LIQUIDS OR 60 °C FOR GASES, USE:

$$Nu_D = 0.027\, Re_D^{0.8}\, Pr^{1/3} \left(\frac{\mu_b}{\mu_w} \right)^{0.14}$$

  - ALL FLUID PROPERTIES ARE CALCULATED AT THE BULK FLUID TEMPERATURE, EXCEPT μ_w WHICH IS EVALUATED AT THE WALL TEMPERATURE.

  - RANGE OF APPLICABILITY:

$$0.7 < Pr < 16{,}700$$

$$Re_D > 10{,}000$$

$$\frac{L}{D} > 60$$

TURBULENT FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

- OTHER CONVECTION HEAT TRANSFER CORRELATIONS FOR TURBULENT FLOWS IN PIPES.

  - THE FOLLOWING CORRELATION APPLIES TO ROUGH WALL PIPES (QUITE ACCURATE):

$$Nu_D = \frac{Re_D \, Pr}{X}\left(\frac{f}{8}\right)\left(\frac{\mu_b}{\mu_w}\right)^n$$

$$X = 1.07 + 12.7\left(Pr^{2/3} - 1\right)\left(\frac{f}{8}\right)^{1/2}$$

- FOR LIQUIDS:

  $n = 0.11$ FOR HEATING,

  $n = 0.25$ FOR COOLING.

- FOR GASES: $n = 0$.

- RANGE OF APPLICABILITY:

$$10^4 < Re_D < 5.\times 10^6$$

$$2 < Pr < 140 \qquad \sim 5\% \;\; Error$$

$$0.5 < Pr < 2{,}000 \qquad \sim 10\% \;\; Error$$

$$0.08 < \frac{\mu_b}{\mu_w} < 40$$

- ALL PHYSICAL PROPERTIES, EXCEPT μ_w ARE EVALUATED AT THE FLUID BULK TEMPERATURE.
- μ_w IS EVALUATED AT THE WALL TEMPERATURE.

- f IS DETERMINED BY USING AN AD HOC CORRELATION.

TURBULENT FLOWS IN DUCTS - HEAT TRANSFER COEFFICIENT

- THE CORRELATIONS OBTAINED FOR CIRCULAR TUBES ON:

  - FRICTION FACTORS,
  - FRICTIONAL PRESSURE GRADIENT, AND
  - CONVECTION HEAT TRANSFER COEFFICIENT

  CAN BE APPLIED TO NON CIRCULAR TUBES BY REPLACING THE DIAMETER (D) APPEARING IN THESE CORRELATIONS BY THE HYDRAULIC DIAMETER DEFINED AS:

$$D_h = \frac{4 \times \text{FLOW SECTION}}{\text{WETTED PERIMETER}} = \frac{4A}{P}$$

► FOR EXAMPLE, THE HYDRAULIC DIAMETER OF AN ANNULAR FLOW SECTION WITH INNER DIAMETER D_1 AND OUTER DIAMETER D_2 IS:

$$D_h = \frac{4\frac{\pi}{4}\left(D_2^2 - D_1^2\right)}{\pi\left(D_2 + D_1\right)} = D_2 - D_1$$

END CONVECTION